Price Discovery in a Matching and Bargaining Market with Aggregate Uncertainty

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Abstract

We introduce aggregate uncertainty into a Rubinstein and Wolinsky (1985)-type dynamic matching and bilateral bargaining model. The market can be either in a high state, where there are more buyers than sellers, or in a low state, where there are more sellers than buyers. Traders do not know the state. They randomly meet each other and bargain by making take-it-or-leave-it offers. The only information transmitted in a meeting is the time a trader spent on the market. There are two kinds of search frictions: time discounting and exogenous exit. We find that as the search frictions vanish, the market discovers the competitive price quickly: the prices offered in equilibrium converge in expectation to the true-state Walrasian price at the rate linear in the total search friction. This rate is the same as it would be if the state were commonly known.

Keywords: Dynamic matching and bargaining, convergence to perfect competition, aggregate uncertainty

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1 Introduction

In dynamic markets with search frictions and decentralized bilateral bargaining, it is common that the aggregate demand and supply conditions are not fully known to the market

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participants. However, such information is important for the bargaining between buyers and sellers and, more generally, for the price discovery process in such a market. For example, in second-hand housing markets, labor markets, etc., traders often care about whether the market they are participating in is a “seller’s market” (i.e., with high demand and low supply) or a “buyer’s market” (i.e., with high supply and low demand). The price offers they make during bargaining and their reservation prices crucially depend on their perceptions of the aggregate market conditions. But in almost all existing work in the literature of dynamic matching and bargaining games, traders are assumed to be certain about the market demand and the market supply.[1]

In a search market where buyers and sellers are strategic and uncertain about the true state of demand and supply, at what price should they trade? This paper studies a simple search market with aggregate uncertainty, in which no individual trader knows the true state of demand and supply but they keep learning about it from their search experiences. We show that, as frictions vanish, buyers and sellers will trade at prices that are approximately competitive (or market clearing, or Walrasian).

Our model is based on a continuous-time variant of Rubinstein and Wolinsky (1985, hereafter RW) and Gale (1987). As in RW, the buyers and sellers are homogeneous: all buyers assign the same value, 1, to the good, and all sellers have the same cost, 0, of providing the good. As in Gale (1987) (and unlike in RW), the inflow rates of buyers and sellers are exogenous. More specifically, buyers and sellers continuously flow into the market, engage in search, get matched pairwise, and bargain for trade. Unlike in RW and Gale (1987), the market can be in one of two states: the high state and the low state; the realized true state is unknown to the traders and unchanged over time. In the high state, the inflow rate of buyers is higher than that of sellers, while the reverse is true in the low state. So the Walrasian price is 1 in the high state, and 0 in the low state.

The market is in a steady state: the inflows and outflows are balanced, keeping the (endogenous) masses of buyers and sellers in the market constant over time. The rate at which buyer-seller pairs are formed depends on the steady-state stocks of buyers and sellers and an exogenous matching function. The traders on the short side are matched with partners at an exogenous Poisson finding rate, while those on the long side at a lower rate depending on the market tightness. The outflows of traders consist of the matched pairs whose bargaining results in trade, and the unmatched traders who exit for exogenous

[1] An important exception is Lauermann et al. (2018), who asked the same convergence question under aggregate uncertainty as ours. See the literature review below.

reasons. Search is costly due to two reasons: time discounting at rate $r \geq 0$ and exogenous exits at rate $\delta > 0$. Our search frictions are thus parameterized by the rates r and δ. Our convergence result pertains to the limiting case where the friction profile (r, δ) tends to $(0, 0)$, or equivalently the total friction level $r + \delta$ tends to 0.

Once a buyer and a seller find each other, they bargain under a random-proposer protocol: one trader is chosen randomly as the proposer who makes a take-it-or-leave-it price offer to the other. If the bargaining is successful (i.e., the offer is accepted), the traders leave the market forever; otherwise, they return to the pool of unmatched traders. We restrict attention to *full trade* market equilibria, i.e., steady-state market equilibria in which bargaining on the equilibrium path always results in trade.

Of course, whether the full trade property is compatible with the bargainers' strategic behaviors depends on their beliefs about the state and their information about each other. We assume that all traders have common prior beliefs concerning the state. Once in the market, they keep updating their beliefs, mainly from their search experiences. Upon finding a partner, every trader observes his partner's market time, i.e., for how long his partner has been participating in the market, but not any other elements of the partner's history. A few words regarding this assumption of information transmission are in order. First, in markets characterized by search, e.g., real estate or labor markets, the information about partners' market times is often available. In the real estate market, for example, buyers are often able to observe for how long the property has been on the market, while sellers may also know, e.g. from their realtors, for how long the buyer has searched. In labor markets, workers may know for how long the position they are applying for has been open, and firms likewise may know for how long the worker has searched. Besides, we will claim (in Remark 2) that at least some information transmission between the meeting partners is needed for our existence and convergence results. In the Online Appendix, we also show that, if we assume traders only observe imperfect signals about their partners' market times, our main results still hold, although the analysis would become much more complex mainly because, unlike in the main text, asymmetric information bargaining occurs even on the equilibrium path.

We show the existence and uniqueness of the full trade equilibrium under every total friction level.[2] Moreover, the full trade equilibrium has several intuitive properties. For

[2]The uniqueness result is more general than the existence result. While we are able to show the full trade equilibrium is unique under every friction profile (r, δ), we are only able to show that, under every total friction level $r + \delta$, a full trade equilibrium exists if the exit rate δ is sufficiently small relative to the

example, the search value (i.e., the expected continuation payoff while being unmatched) of a trader is decreasing in his own market time. As the trader spends more time searching, he gets more pessimistic about the state, which worsens his prospects of future gains.[3] Furthermore, relative to the benchmark case in which the state is commonly known, the equilibrium prices involve an uncertainty discount if the state is high, and an uncertainty premium if it is low. This is intuitive: when, for example, the state is high, traders put some probability on the low state, where the competitive price is low. The offers that they make reflect this and are below the high-state certainty price.

A market equilibrium features a distribution of prices. This is because traders spend different times searching on the market and update differently, and hence in different meetings traders will have different beliefs concerning the state. *Our main result is that, nevertheless, most transactions occur at prices that are approximately Walrasian when search frictions are small.* More precisely, we show that as $(r, \delta) \to (0, 0)$, the prices offered in equilibrium converge in expectation to 1 if the state is high and to 0 if the state is low, at the rate linear in the total friction $r + \delta$. This rate of convergence is the same as it would be if the state were commonly known.

The convergence can be roughly understood as follows. Suppose the state is high so that there are more buyers than sellers flowing into the market. Since traders leave the market either through trade, i.e., in matched pairs, or for exogenous reasons, in the steady state we must have more buyers in the market; that is, buyers are on the long side and sellers are on the short side. In fact, as the exogenous exit rate vanishes, buyers will accumulate indefinitely while the stock of sellers in the market will be bounded. Under these conditions, most buyers would find it difficult to find sellers, while sellers spend relatively little time searching in the market on average. In most instances, immediately before a buyer and a seller meet each other, the buyer's belief is rather accurate, while the seller's belief is not "too wrong." After observing each other's market time, their beliefs concerning the state will adjust to the same level. Effectively, the long side's information (here the buyer's), which is rather precise, is transmitted to the poorly informed short side (here the seller), and this results in the precise common belief concerning the state. Given this common belief, bargaining transpires under symmetric information. Thus, it

discount rate r. But our existence result is satisfactory for the sake of supporting our convergence result, because our convergence result works for any vanishing path of friction profile (r, δ), regardless of how the composition of frictions changes along the path. See also footnote 4.

[3] Throughout the paper we use feminine pronouns for sellers and masculine pronouns for buyers and traders.

seems plausible to conjecture that, if they trade, the transaction price would be close to the true-state certainty benchmark prices.[4] Moreover, it is now well known from the literature that the true-state certainty prices are approximately the true-state Walrasian price when both the discount rate and the exogenous exit rate are small.

With the above being said, the rigorous argument for our convergence result is far more involved for at least two reasons. First, our convergence result also provides the exact rate of convergence. Second, and more importantly, in order to prove the conjecture that when frictions are small the equilibrium transaction prices are close to the true-state certainty benchmark prices, we have to deal with off-equilibrium and unboundedly higher-order beliefs (i.e., beliefs about others' beliefs, beliefs about others' beliefs about others' beliefs, and so on) concerning the state. Indeed, the bargaining behaviors in an equilibrium-path meeting depend on the buyer's and the seller's outside option values, which are their search values should the bargaining break down. Such search values depend on the traders' prospects of meeting and bargaining with future partners, which in turn depend on the outside option values in the future meetings, which in turn depend on the future partners' prospects of meeting and bargaining with further future partners, ... and so on. Thus, off-equilibrium search values and beliefs, both in unboundedly high order, are involved although we are originally concerned with only the equilibrium-path behaviors.

Roughly speaking, we overcome the above "problem of infinite regress" and manage to prove our rate of convergence result by (i) bounding the imprecision of a large class of off-equilibrium beliefs concerning the state, and (ii) recursively and geometrically bounding the expected discrepancy between off-equilibrium search values and their no-uncertainty counterparts. From (ii) we can show that the expected discrepancy between the outside option values for equilibrium meetings and their no-uncertainty counterparts is bounded by a geometric-type series of the imprecision of the first- and progressively higher-order beliefs concerning the state; then (i) allows us to show that this series, after being divided by the friction level, is convergent. Hence, the outside option values for equilibrium meetings have an upper bound proportional to the friction level.

[4] Although the bargaining in every meeting on the equilibrium path transpires under symmetric information, it is not immediate that every meeting on the equilibrium path results in trade. To verify the full trade property, we need to show that if a pair breaks down and both traders search for another partners off the equilibrium path, their total value of outside options (i.e., sum of search values) does not exceed 1. As it turns out, showing this is highly nontrivial because bargaining off the equilibrium path would not be under a common belief, but under a very complicated belief structure.

Literature There is a voluminous literature on dynamic matching and bargaining games, including RW, Gale (1987); Wolinsky (1988); Satterthwaite and Shneyerov (2007, 2008); Atakan (2008, 2009); Shneyerov and Wong (2010a,b, 2011); Lauermann (2013); Lauermann et al. (2018); Majumdar et al. (2016). Most of it, however, assumes the market demand and supply are common knowledge.

Our benchmark model (in which the state of the market is commonly known) is a variant of RW, who assume the stocks of buyers and sellers in the market are exogenously given, while each pair that leaves through trade gets replaced by a clone. The latter assumption is imposed to maintain the exogenous stocks. RW's main result is that the equilibrium prices are *not* Walrasian in the limit.[5] There are two important and interrelated differences between their model and ours. First, unlike in RW (but as in Gale (1987)), in our model flows are exogenous, while the stocks are endogenous. The second difference is that we assume exogenous exits as in Satterthwaite and Shneyerov (2008) and Lauermann (2013). This assumption ensures that we have a steady state even though the incoming flows are unbalanced. But the unbalanced nature of the flows results in the stocks that are highly unbalanced. In fact, the ratio of the short and long side stocks vanishes as the exogenous exit rate tends to 0. In contrast, this ratio is constant in RW. (If, on the other hand, we formally consider limit prices in RW as the ratio of the stocks vanishes, then prices also become Walrasian in the limit in their model, as they do in ours.)

Very few papers have studied aggregate uncertainty in dynamic matching markets, and even less is known about the rates of convergence. Wolinsky (1990) considers a model with a related but different kind of uncertainty, known as *common value* uncertainty. While the uncertainty in our model is about the relative scarcity of the good, in his model the uncertainty is about the common consumption value or cost of the good, and he obtains essentially negative results concerning convergence. Under common value uncertainty, it turns out that buyers and sellers prefer to experiment for too long and insist on bargaining positions too far apart compared to the competitive setting. In addition, the range of bargaining positions is restricted to two, which is another obstacle to convergence. Subsequent work by Serrano (2002) and Blouin and Serrano (2001) demonstrates that this difficulty is robust. In our model, on the other hand, buyers and sellers can make offers at any price

[5] The Walrasian price in RW is defined by stock demand and stock supply. However, as clarified by Gale (1987), the "correct" Walrasian benchmark price in such a search model should be defined by flow demand and flow supply instead. Under RW's cloning assumption, the equilibrium price in RW (in fact, any price between the sellers' value 0 and the buyers' value 1) is actually "flow Walrasian."

they like, and have private values.

The work most related to ours is Lauermann et al. (2018), who consider a private value setting like ours and obtain positive convergence results. The main differences between their setting and ours are in the matching technology, the trading protocol, and the information transmission within a meeting. While we assume pairwise matching and random-proposer bargaining, they, in a discrete-time model, assume many-to-one matching and auctions with secret reserve price as in Satterthwaite and Shneyerov (2007, 2008); so even buyers on the long side always get a match in every period. They assume no information transmission before an auction and minimal informational feedback after an auction.[6] The different settings lead to very different forces driving the convergence towards the competitive outcome. In their model, there is no such thing as learning from search; the convergence is driven by the competition among bidders and the "loser's curse" as in Pesendorfer and Swinkels (1997): when the true state is high, losing bidders become more and more pessimistic over time, and bid higher. In our model, there are no competition among bidders and loser's curse; the convergence is instead driven by the learning from search and the information transmission from the long to the short side.

There is also a recent paper by Majumdar et al. (2016) who address essentially the same question, but in a very different model. Their model involves matching and bargaining of buyers and sellers that have heterogeneous values and costs. Learning transpires without a common prior; rather, it involves optimistic priors that put all the weight on the state most favorable for a trader. The belief updating is not in a Bayesian manner.

In terms of the rates of convergence, Shneyerov and Wong (2010b) and Lauermann (2012) also find a linear convergence rate. Both papers consider dynamic matching in a private information setting. The first paper considers bargaining, while the second assumes one-sided price offers. Our finding reiterates the message that it is the time consuming search that manifests itself as the main friction in dynamic matching markets, while other elements such as private information or aggregate uncertainty, do not slow the speed of convergence.

The novelty of our paper is to show the convergence to perfect competition in a market with aggregate uncertainty and pairwise bargaining, within the standard common prior and Bayesian updating paradigm, and derive the exact rate of convergence.

[6] It means, traders who failed to trade in an auction only know the failure. In particular, they do not observe the number of bidders (even the seller do not know whether there is any bidder at all), the reserve price and bids after the auction.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 formulates the full trade market equilibrium. Section 4 develops the uniqueness of the full trade equilibrium and shows some basic equilibrium properties. Section 5 states and outlines the proof of our main convergence result. Section 6 develops the existence of the full trade equilibrium. Section 7 concludes. Proofs are relegated to the Appendix. The Online Appendix extends our model by assuming traders only observe imperfect signals about their partners' market times and shows that our main results are robust to this extension.

2 Model

There are a continuum of buyers and sellers trading an indivisible, homogeneous good in a decentralized market, which operates in continuous time with infinite horizon. Each buyer demands one unit of the good and values it at 1. Each seller has one unit of the good and values it at 0. Buyers and sellers arrive at the market deterministically and continuously over time, at constant inflow rates depending on the state of the market. The market can be in one of two states, $\omega = H$ or $\omega = L$, which is unchanged over time. The true state is unknown to the traders. The common prior belief that the state is ω is denoted as $\phi^\omega \in [0, 1]$. The inflow rates of buyers and sellers in state ω are denoted as $\lambda_B^\omega > 0$ and $\lambda_S^\omega > 0$, respectively. (That is, in state ω, new buyers (sellers) of mass λ_B^ω (λ_S^ω) arrive at the market per unit time.) We assume that in state H, the inflow rate of the buyers is higher than that of the sellers, and the opposite holds in state L, i.e.,

$$\lambda_B^H > \lambda_S^H > 0, \qquad \lambda_S^L > \lambda_B^L > 0. \tag{1}$$

Note that, under this assumption, the Walrasian price (in the flow sense) is 1 if $\omega = H$ and 0 if $\omega = L$.

Upon arrival, each trader starts searching for a trading partner. The process of search is modeled as the following random matching process. Active buyers and sellers are randomly matched into buyer-seller pairs at the aggregate flow rate given by $\mu \cdot \min\{\Lambda_B, \Lambda_S\}$, where Λ_B and Λ_S are the masses of active buyers and active sellers currently in the market, respectively.[7] (That is, given the current stock Λ_B of buyers and Λ_S of sellers, the mass of

[7] This special "minimum-form matching function" is not crucial to our results, but allows us to explicitly express the steady state in terms of parameters and simplify the exposition. See Remark 1 below.

buyer-seller pairs matched per unit time is $\mu \cdot \min\{\Lambda_B, \Lambda_S\}$.)

Once a pair of buyer and seller is matched, they bargain under the information that we describe below in Subsection 3.2. The bargaining protocol is as follows. With probability $\beta_B \in (0,1)$, the buyer makes a take-it-or-leave-it price offer; with probability $\beta_S = 1 - \beta_B$, the seller makes a take-it-or-leave-it price offer. If a price offer p is accepted, the pair trades and leaves the market forever, with the buyer's payoff $1-p$ and the seller's payoff p. Otherwise, the pair breaks down, and both traders return to the market pools searching for another match.

We shall focus on steady state throughout. In order to ensure the existence of a steady state, we assume that traders, except through trading, also leave the market for exogenous reasons at the exit rate $\delta > 0$. A trader who exits the market without trading has a zero payoff. Moreover, all traders are risk neutral and discount future at the instantaneous rate $r \geq 0$, so that they maximize the expected discounted payoffs.

The pair $(r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ of the discount and exit rates is the *friction profile* of our model.[8] We also call $r + \delta$ the total friction level, or simply *friction level.* Our asymptotic results will pertain to the case of vanishing frictions: $(r, \delta) \to (0, 0)$, or equivalently $r + \delta \to 0$.

3 Equilibrium

Throughout this paper we consider steady-state equilibria in which every meeting on the equilibrium path results in trade, which we call *full trade (market) equilibria* for short. The formal definition of full trade equilibria will be given in Subsection 3.3. Conceptually, a steady-state equilibrium consists of steady-state stocks and distributions of traders, traders' beliefs about the state, and traders' bargaining strategies such that (i) given the traders' bargaining strategies, the inflows and outflows of traders are balanced so that the stocks and distributions of traders are maintained at their steady-state values, and (ii) given the steady-state stocks and distributions, the traders' beliefs and bargaining strategies constitute a perfect Bayesian equilibrium. In addition, we restrict attention to those steady-state equilibria that have full trade.[9]

Focusing on full trade equilibria allows us to recursively determine the three kinds of

[8] Although the discount rate and the exit rate are assumed to be common for buyers and sellers, our analysis can be easily extended to different discount and exit rates.

[9] Starting from Subsection 3.2 we will also restrict attention to equilibria with "passive beliefs."

equilibrium objects. Specifically, deriving the steady-state stocks and distributions does not rely on the beliefs and the bargaining strategies. Given the steady-state distributions, deriving the beliefs (on and off the equilibrium path) does not rely on the bargaining strategies. Having determined the steady-state distributions and the beliefs, we can finally tackle the most challenging objects, the bargaining strategies.

3.1 Steady state

Let $F_B^\omega(t_B)$ denote the fraction of buyers who have been in the market for less than time t_B, out of the market stock of buyers, in the steady state in state ω. Similarly let $F_S^\omega(t_S)$ denote the analogous fraction for sellers. Also let $f_B^\omega(t_B)$ and $f_S^\omega(t_S)$ denote the corresponding densities, respectively.

For each state $\omega \in \{L, H\}$ and for any $t_B \geq 0$, to maintain the measure $\Lambda_B^\omega F_B^\omega(t_B)$ of buyers with market time less than t_B, the following steady state equation has to hold:

$$\lambda_B^\omega = \delta \Lambda_B^\omega F_B^\omega(t_B) + \mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\} F_B^\omega(t_B) + \Lambda_B^\omega f_B^\omega(t_B). \tag{2}$$

The LHS of (2) is the rate of inflows due to the arrival of new buyers. The three terms on the RHS of (2) are the rates of outflows due to (i) exogenous exit, (ii) getting matched, and (iii) accumulation of market times for those unmatched.

Similarly, the steady state equation for sellers is, for $\omega \in \{L, H\}$ and $t_S \geq 0$,

$$\lambda_S^\omega = \delta \Lambda_S^\omega F_S^\omega(t_S) + \mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\} F_S^\omega(t_S) + \Lambda_S^\omega f_S^\omega(t_S). \tag{3}$$

In particular, if we consider the whole pools, the steady state equations are

$$\lambda_B^\omega = \delta \Lambda_B^\omega + \mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}, \tag{4}$$

$$\lambda_S^\omega = \delta \Lambda_S^\omega + \mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}. \tag{5}$$

Given the stocks $\Lambda_B^\omega, \Lambda_S^\omega$, the Poisson finding rates $\alpha_B^\omega, \alpha_S^\omega$ at which buyers and sellers meet their partners are

$$\alpha_B^\omega = \frac{\mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}}{\Lambda_B^\omega}, \quad \alpha_S^\omega = \frac{\mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}}{\Lambda_S^\omega}. \tag{6}$$

The steady-state finding rates $\alpha_B^\omega, \alpha_S^\omega$ and distributions F_B^ω, F_S^ω can be solved from the above steady state equations.

Lemma 1. *In the steady state,*

$$\alpha_B^L = \alpha_S^H = \mu, \tag{7}$$

$$\alpha_B^H = \frac{\delta\mu\lambda_S^H}{(\delta+\mu)\lambda_B^H - \mu\lambda_S^H} < \mu, \tag{8}$$

$$\alpha_S^L = \frac{\delta\mu\lambda_B^L}{(\delta+\mu)\lambda_S^L - \mu\lambda_B^L} < \mu, \tag{9}$$

$$F_B^\omega(t_B) = 1 - \exp(-(\delta+\alpha_B^\omega)t_B) \quad \textit{for } \omega \in \{L,H\}, t_B \geq 0, \tag{10}$$

$$F_S^\omega(t_S) = 1 - \exp(-(\delta+\alpha_S^\omega)t_S) \quad \textit{for } \omega \in \{L,H\}, t_S \geq 0. \tag{11}$$

Moreover, the steady-state long-side finding rates α_B^H and α_S^L given by (8) and (9) are at most of order δ; more precisely,

$$\alpha_B^H \leq \frac{\lambda_S^H}{\lambda_B^H - \lambda_S^H} \cdot \delta, \tag{12}$$

$$\alpha_S^L \leq \frac{\lambda_B^L}{\lambda_S^L - \lambda_B^L} \cdot \delta. \tag{13}$$

Remark 1. Although the exact expressions for the finding rates in (7)–(9) rely on our "minimum-form matching function," those for the distributions in (10) and (11) do not. Instead of assuming the minimum-form, we could also assume a nondecreasing, concave, and homogeneous of degree 1 matching function $M(\Lambda_B, \Lambda_S)$ such that $\lim_{\Lambda_S \to 0} M(\Lambda_B, \Lambda_S) = 0$ for every $\Lambda_B > 0$ and $\lim_{\Lambda_B \to 0} M(\Lambda_B, \Lambda_S) = 0$ for every $\Lambda_S > 0$. Under these conditions it can be shown that (i) the finding rates $\alpha_B(\zeta) \equiv M(1, 1/\zeta)$ for buyers and $\alpha_S(\zeta) \equiv M(\zeta, 1)$ for sellers depend only on the market tightness $\zeta \equiv \Lambda_B/\Lambda_S$; (ii) for any $\delta > 0$ and each state ω there is a unique steady state; and (iii) the steady-state market tightness ζ^ω ($\omega = L, H$) satisfies $\zeta^L < 1 < \zeta^H$, $\lim_{\delta\to 0} \zeta^L = 0$, and $\lim_{\delta \to 0} \zeta^H = \infty$. If we further assume that $\alpha_B(\zeta^H)$ is of order $1/\zeta^H$ as $\zeta^H \to \infty$ and $\alpha_S(\zeta^L)$ is of order ζ^L as $\zeta^L \to 0$, then it can be shown that $\alpha_B(\zeta^H)$ and $\alpha_S(\zeta^L)$ are of order δ as $\delta \to 0$ and our analysis can be adapted to show that all of the main results still hold.

In state ω, for those who are currently unmatched, the probability of ultimately being matched (rather than exogenously exiting before being matched) is

$$m_B^\omega \equiv \frac{\alpha_B^\omega}{\delta + \alpha_B^\omega}, \qquad m_S^\omega \equiv \frac{\alpha_S^\omega}{\delta + \alpha_S^\omega} \tag{14}$$

for buyers and sellers, respectively. Using (7)–(9), we have

$$m_B^L = m_S^H = \frac{\mu}{\delta + \mu},$$

$$m_B^H = \frac{\mu \lambda_S^H}{(\delta + \mu)\lambda_B^H}, \quad m_S^L = \frac{\mu \lambda_B^L}{(\delta + \mu)\lambda_S^L}.$$

Notice that, other than being interpreted as steady state distributions, $F_B^\omega(t_B)$ and $F_S^\omega(t_S)$ have several other interpretations. From an unmatched buyer's point of view and conditional on state ω, $F_B^\omega(t_B)$ can be interpreted as (i) the probability that he will leave the market (due to either trade or exogenous exit) after some searching time less than t_B; or (ii) the probability of being matched after some searching time less than t_B conditional on the event that he will ultimately be matched at all (rather than exogenously exit before being matched);[10] or (iii) the probability of exiting exogenously after some searching time less than t_B conditional on the event that he will exogenously exit at all. The interpretation (ii) above will be particularly useful in our analysis. Of course, $F_S^\omega(t_S)$ can be interpreted analogously from an unmatched seller's point of view.

3.2 Beliefs

Now we consider the traders' beliefs concerning the state of the market. Since there are only two possible states and every trader updates his belief about the two possible states according to the Bayes' rule as his market experiences accumulate, it will prove convenient to work with the trader's *optimism level*, defined as the ratio of the belief probability of the trader's favorable state (which is L for buyers and H for sellers) to that of his unfavorable state.

Recall that the common prior belief about state $\omega \in \{L, H\}$ is $\phi^\omega \in [0, 1]$. So the

[10] In state ω, the probability of a currently unmatched buyer being matched after further searching time within $[t, t + dt]$ is $e^{-\left(\delta + \alpha_B^\omega\right)t}\alpha_B^\omega dt$. Therefore, the probability of being matched after some searching time less than t_B is

$$\int_0^{t_B} e^{-\left(\delta + \alpha_B^\omega\right)t}\alpha_B^\omega dt = \frac{\alpha_B^\omega}{\delta + \alpha_B^\omega}\left(1 - e^{-\left(\delta + \alpha_B^\omega\right)t_B}\right) = m_B^\omega F_B^\omega(t_B).$$

prior optimism levels are ϕ^L/ϕ^H for buyers and ϕ^H/ϕ^L for sellers. But the mere fact that a trader is born as a buyer or as a seller conveys information about the state because inflow rates are different across states. Specifically, upon arrival, buyers' and sellers' initial optimism levels are, respectively,[11]

$$\xi_B^0 \equiv \frac{\phi^L \lambda_B^L}{\phi^H \lambda_B^H}, \qquad \xi_S^0 \equiv \frac{\phi^H \lambda_S^H}{\phi^L \lambda_S^L}. \tag{15}$$

We now make the following assumption concerning the information transmission once a buyer and a seller meet each other.

Assumption 1 (Information transmission)**.** *Upon meeting, each trader observes the total time his or her partner has participated in the market.*

Assumption 1 ensures that in every meeting on the equilibrium path (which is the first one for both traders provided the equilibrium is full trade), the buyer and the seller have symmetric information. However, in a meeting off the equilibrium path, at least one trader would have private information, as only his total market time rather than his full market history would be observable to his partner.[12]

Under Assumption 1, we can show (in Section 6) the existence of a full trade equilibrium for every friction level. We will also claim (in Remark 2) that, if we assume traders do not observe any histories of their partners upon meeting, then full trade equilibrium does not exist at least when frictions are sufficiently small. (A quick intuition is that some meetings would have both the buyer and the seller being so optimistic towards their own favorable states.) So at least some information transmission between the meeting partners is needed for our convergence result based on full trade equilibria. In the Online Appendix, we also show that our main results would still hold if we assume traders only observe imperfect signals about their partners' market times, although the analysis would become much more complex mainly because asymmetric information bargaining would then occur even on the equilibrium path.

[11] Our main results would still hold if both buyers and sellers, upon arrival, believe that the probability of state ω is ϕ^ω.

[12] Our information transmission is less than that assumed in the recent information percolation literature, notably Duffie and Manso (2007) and Duffie et al. (2009). There, it is assumed that traders can observe *beliefs* of each other, or equivalently, traders can observe each other's full histories. Given that we restrict attention to equilibria where each meeting results in trade, our assumption is equivalent to theirs on the equilibrium path, but much weaker than theirs off the equilibrium path.

How do traders update their beliefs about the state according to their market histories? In general, a trader's market history (on or off the equilibrium path) consists of his search experiences and bargaining experiences. Consider a buyer who is currently in his n-th meeting, immediately before the bargaining in that meeting begins. His search experiences are characterized by his *search history* of the form $(t_{B1}, \ldots, t_{Bn}; t_{S1}, \ldots, t_{Sn})$, where $t_{Bi} \geq 0$ for $i \in \{1, \ldots, n\}$ is the search time the buyer has spent to have his i-th meeting, and $t_{Si} \geq 0$ for $i \in \{1, \ldots, n\}$ is the observed time on the market of the i-th seller he has met. Similarly, the search history of a seller immediately before the bargaining in her n-th meeting is of the form $(t_{S1}, \ldots, t_{Sn}; t_{B1}, \ldots, t_{Bn})$, where $t_{S1}, \ldots, t_{Sn} \geq 0$ are the seller's search times, and $t_{B1}, \ldots, t_{Bn} \geq 0$ are her partners' times on the market.[13]

On the other hand, a trader's *bargaining history* includes all the information he has learned from his bargaining experiences, i.e., which side proposed in any past and current meetings, every price offer received from the partners in those meetings, every price offers previously made by himself, and the fact that all the offers in his previous meetings are rejected.

3.2.1 No updating from bargaining history

Now we make the assumption of passive beliefs, which is common in the literature.

Assumption 2 (Passive beliefs)**.** *Traders do not update beliefs from off-the-equilibrium-path observations on partners' actions.*

Under Assumptions 1 and 2 and given that we focus on full trade equilibria, we claim that *traders (on or off the equilibrium path) only use their search histories but not bargaining histories to update their beliefs concerning the state.*

To see this, first note that, in a full trade equilibrium, the transaction price of every equilibrium-path meeting depends only on (i) whether the buyer or the seller makes offer, and (ii) the realized search times t_B, t_S that the buyer and the seller have spent to find each other. So let $p_B(t_B, t_S)$ denote the buyers' equilibrium offer and $p_S(t_S, t_B)$ denote the sellers' equilibrium offer. Under Assumption 1 both the proposer and the responder in such a meeting observe t_B, t_S and understand what price is supposed to be proposed.

[13] Despite our focus on full trade equilibria (where every trader participates in at most one meeting in his whole life), it is necessary for our purpose to consider general search histories involving $n > 2$ meetings, due to the problem of infinite regress that we mention in the Introduction and Section 5.

Now consider a (possibly off-equilibrium) trader whose bargaining history is nonempty. No matter how many partners this trader has met, in a full trade equilibrium he believes that he is the first partner of his partners. The information from this trader's bargaining history can be decomposed into two classes. One includes those pieces of information that do not surprise him, e.g., his own past (on or off the equilibrium path) actions, which side proposed in any past and current meetings, that a previous partner proposed an equilibrium offer (which was rejected by him), that the current partner proposed an equilibrium offer (waiting for his response), and that he proposed a price that should be rejected in equilibrium and was actually rejected. According to the Bayes' rule the trader does not update beliefs from this class of non-surprising information. The other class includes those pieces of information that do surprise him, e.g., a received price offer that should not be received in equilibrium, and that a previous partner rejected a price offer that should not be rejected in equilibrium. By Assumption 2 the trader does not update beliefs from this class of surprising information either.

3.2.2 Updating from search history

Every time a buyer meets a seller after spending time t_B, by Bayes' rule his optimism level is updated by multiplying the following factor: (see footnote 10)

$$\xi_B^{T_B}(t_B) \equiv \frac{m_B^L f_B^L(t_B)}{m_B^H f_B^H(t_B)} = \frac{\alpha_B^L \exp(-\alpha_B^L t_B)}{\alpha_B^H \exp(-\alpha_B^H t_B)}, \tag{16}$$

and if the observed market time of the partner seller is t_S, his optimism level is further updated by multiplying the following factor:

$$\xi_B^{T_S}(t_S) \equiv \frac{f_S^L(t_S)}{f_S^H(t_S)} = \frac{(\delta + \alpha_S^L)\exp(-\alpha_S^L t_S)}{(\delta + \alpha_S^H)\exp(-\alpha_S^H t_S)}. \tag{17}$$

Hence, for a buyer immediately before the bargaining in his n-th meeting, with search history $(t_{B1}, \ldots, t_{Bn}; t_{S1}, \ldots, t_{Sn})$, his current optimism level after Bayesian updating is

$$\begin{aligned} &\xi_B^0 \xi_B^{T_B}(t_{B1}) \xi_B^{T_S}(t_{S1}) \cdots \xi_B^{T_B}(t_{Bn}) \xi_B^{T_S}(t_{Sn}) \\ &\qquad = \xi_B^0 \cdot \left(\frac{\alpha_B^L(\delta + \alpha_S^L)}{\alpha_B^H(\delta + \alpha_S^H)} \right)^n \cdot \frac{\exp(-\alpha_B^L \sum_{i=1}^n t_{Bi} - \alpha_S^L \sum_{i=1}^n t_{Si})}{\exp(-\alpha_B^H \sum_{i=1}^n t_{Bi} - \alpha_S^H \sum_{i=1}^n t_{Si})}, \end{aligned}$$

which depends on $(t_{B1}, \ldots, t_{Bn}; t_{S1}, \ldots, t_{Sn})$ only through n, $t_B = \sum_{i=1}^n t_{Bi}$, and $t_S = \sum_{i=1}^n t_{Si}$. So, for $n = 1, 2, \ldots$ and $t_B, t_S \geq 0$, we define

$$\xi_B^n(t_B, t_S) \equiv \xi_B^0 \cdot \left(\frac{\alpha_B^L(\delta + \alpha_S^L)}{\alpha_B^H(\delta + \alpha_S^H)} \right)^n \cdot \frac{\exp(-\alpha_B^L t_B - \alpha_S^L t_S)}{\exp(-\alpha_B^H t_B - \alpha_S^H t_S)} \tag{18}$$

to mean the optimism level of such a buyer with t_B being his own total market time and t_S being the total market times of all the partners he has met.

Similarly, the updating factors for a seller's optimism level, after searching for time t_S and observing a partner's market time t_B, are:

$$\xi_S^{T_S}(t_S) \equiv \frac{m_S^H f_S^H(t_S)}{m_S^L f_S^L(t_S)} = \frac{\alpha_S^H \exp(-\alpha_S^H t_S)}{\alpha_S^L \exp(-\alpha_S^L t_S)}, \tag{19}$$

$$\xi_S^{T_B}(t_B) \equiv \frac{f_B^H(t_B)}{f_B^L(t_B)} = \frac{(\delta + \alpha_B^H) \exp(-\alpha_B^H t_B)}{(\delta + \alpha_B^L) \exp(-\alpha_B^L t_B)}. \tag{20}$$

For $n = 1, 2, \ldots$ and $t_S, t_B \geq 0$, we define

$$\xi_S^n(t_S, t_B) \quad \equiv \quad \xi_S^0 \cdot \left(\frac{\alpha_S^H(\delta + \alpha_B^H)}{\alpha_S^L(\delta + \alpha_B^L)} \right)^n \cdot \frac{\exp(-\alpha_S^H t_S - \alpha_B^H t_B)}{\exp(-\alpha_S^L t_S - \alpha_B^L t_B)}, \tag{21}$$

with the analogous interpretation for a seller immediately before the bargaining in her n-th meeting. Since $\alpha_B^L > \alpha_B^H$ and $\alpha_S^L < \alpha_S^H$, it is clear that $\xi_B^{T_B}$, $\xi_S^{T_S}$ are strictly decreasing; $\xi_B^{T_S}$, $\xi_S^{T_B}$ are strictly increasing; for every $n = 1, 2, \ldots$, ξ_B^n, ξ_S^n are strictly decreasing in the first argument and strictly increasing in the second argument.

Of course, a trader's optimism can be easily translated into his belief probabilities. Given a current optimism level $\xi \in \overline{\mathbb{R}}_+ \equiv [0, \infty]$, buyers' and sellers' current beliefs about state ω, denoted as $\pi_B^\omega(\xi), \pi_S^\omega(\xi)$ respectively, are given by

$$\pi_B^L(\xi) = \pi_S^H(\xi) = \frac{\xi}{1 + \xi}, \qquad \pi_B^H(\xi) = \pi_S^L(\xi) = \frac{1}{1 + \xi}. \tag{22}$$

Lemma 2. *For any $t_B, t_S \geq 0$, $\pi_B^L(\xi_B^1(t_B, t_S)) = \pi_S^L(\xi_S^1(t_S, t_B))$ and $\pi_B^L(\xi_B^n(t_B, t_S)) > \pi_S^L(\xi_S^n(t_S, t_B))$ when $n = 2, 3 \ldots$.*

Lemma 2 in particular implies that the buyer and the seller in an equilibrium-path meeting (whose optimism levels are given by ξ_B^1 and ξ_S^1 respectively) hold a common belief concerning the state. This is not surprising because under Assumption 1 the traders in such a meeting have symmetric information.

3.3 Bellman equations and full trade equilibrium

The equilibrium bargaining strategies (including proposing strategies and responding strategies) of the buyers and sellers will be fully characterized by their search values.

Let $W_B(\xi, t_B) \in [0,1]$ denote the buyers' *search value*, i.e., the expected continuation payoff of a currently unmatched buyer when his current belief is characterized by an optimism level $\xi \in [0,\infty]$ and his market time is $t_B \geq 0$. Let $V_B(\xi, t_B; t_S) \in [0,1]$ denote the buyers' *match value*, i.e., the expected continuation payoff of a currently matched buyer who has some market time $t_B \geq 0$, and had some optimism level $\xi \in [0,\infty]$ immediately before observing his current partner's market time $t_S \geq 0$. Analogously, let $W_S(\xi, t_S) \in [0,1]$ denote the sellers' search value and $V_S(\xi, t_S; t_B) \in [0,1]$ the sellers' match value. Note that these continuation payoffs could be off the equilibrium path.

Given the search values W_B, W_S, the equilibrium match values V_B, V_S are given by

$$\begin{aligned} V_B(\xi, t_B; t_S) = \beta_B \max\left\{1 - W_S(\xi_S^1(t_S, t_B), t_S), W_B(\xi\xi_B^{T_S}(t_S), t_B)\right\} \\ + \beta_S \max\left\{W_B(\xi_B^1(t_B, t_S), t_B), W_B(\xi\xi_B^{T_S}(t_S), t_B)\right\}, \end{aligned} \tag{23}$$

$$\begin{aligned} V_S(\xi, t_S; t_B) = \beta_S \max\left\{1 - W_B(\xi_B^1(t_B, t_S), t_B), W_S(\xi\xi_S^{T_B}(t_B), t_S)\right\} \\ + \beta_B \max\left\{W_S(\xi_S^1(t_S, t_B), t_S), W_S(\xi\xi_S^{T_B}(t_B), t_S)\right\}. \end{aligned} \tag{24}$$

To understand (23), consider a buyer with optimism level and market time (ξ, t_B) immediately before the current meeting. Once he meets a seller with market time t_S, he observes t_S and believes this meeting is the seller's first one, and updates his optimism level to $\xi\xi_B^{T_S}(t_S)$, so that his search value would be $W_B(\xi\xi_B^{T_S}(t_S), t_B)$ should the bargaining break down. The seller, since she can only observe t_B but not the buyer's full history, will assume (correctly or incorrectly) that the buyer has not participated in any past meeting and that the buyer's search value is $W_B(\xi_B^1(t_B, t_S), t_B)$. On the other hand, the seller herself, after observing t_B, has search value $W_S(\xi_S^1(t_S, t_B), t_S)$. In a perfect Bayesian equilibrium, if the buyer can make a take-it-or-leave-it offer, which occurs with probability β_B, he will either offer $W_S(\xi_S^1(t_S, t_B), t_S)$, which will be accepted, or offer something that will be rejected, leaving the buyer his search value. If the seller can make a take-it-or-leave-it offer, which occurs with probability β_S, she will offer the buyer the payoff level of $W_B(\xi_B^1(t_B, t_S), t_B)$, which may be either higher or lower than the buyer's reservation value $W_B(\xi\xi_B^{T_S}(t_S), t_B)$. In equilibrium the buyer will accept this offer if and only if it gives the buyer at least this reservation value. It explains (23). Of course, (24) can be understood similarly.

Given the match values V_B, V_S, the equilibrium search values W_B, W_S are given by

$$W_B(\xi, t_B) = \sum_\omega \pi_B^\omega(\xi) m_B^\omega \iint e^{-rt'_B} V_B(\xi\xi_B^{T_B}(t'_B), t_B + t'_B; t'_S) dF_B^\omega(t'_B) dF_S^\omega(t'_S), \tag{25}$$

$$W_S(\xi, t_S) = \sum_\omega \pi_S^\omega(\xi) m_S^\omega \iint e^{-rt'_S} V_S(\xi\xi_S^{T_S}(t'_S), t_S + t'_S; t'_B) dF_S^\omega(t'_S) dF_B^\omega(t'_B). \tag{26}$$

To understand (25), consider an unmatched buyer with optimism level and market time (ξ, t_B). If the true state is ω, which he currently believes is the case with probability $\pi_B^\omega(\xi)$, then he ultimately meets a seller with probability m_B^ω. Conditional on this event, his additional search time t'_B is distributed according to $F_B^\omega(\cdot)$, and the market time t'_S of the seller he meets next is distributed according to $F_S^\omega(\cdot)$. Upon meeting such a seller, he will obtain the match value $V_B(\xi\xi_B^{T_B}(t'_B), t_B + t'_B; t'_S)$. It explains (25). The explanation for (26) is parallel.

Now, consider an equilibrium-path meeting. Let the buyer's market time be t_B and the seller's market time be t_S, which are mutually observable by assumption. If the matching surplus $1 - W_B(\xi_B^1(t_B, t_S), t_B) - W_S(\xi_S^1(t_S, t_B), t_S)$ is negative, then this meeting must not result in trade. Suppose $W_B(\xi_B^1(t_B, t_S), t_B) + W_S(\xi_S^1(t_S, t_B), t_S) \leq 1$. If the buyer is the proposer, he will make the price offer

$$p_B(t_B, t_S) = W_S(\xi_S^1(t_S, t_B), t_S),$$

which is just the seller's expected continuation payoff should she reject the offer and continue searching in the market. Similarly, if the seller proposes, she will propose the price offer

$$p_S(t_S, t_B) = 1 - W_B(\xi_B^1(t_B, t_S), t_B)$$

to make the buyer indifferent between accepting or rejecting the offer. Both offers will be accepted on the equilibrium path. So an equilibrium transaction price is either $p_B(t_B, t_S)$ if the buyer proposes, or $p_S(t_S, t_B)$ if the seller proposes.

We view (23)–(26) as definitional for a full trade equilibrium.

Definition 1 (Full trade equilibrium)**.** A *full trade (market) equilibrium* is a tuple (W_B, W_S, V_B, V_S) such that $W_B, W_S : \overline{\mathbb{R}}_+ \times \mathbb{R}_+ \to [0,1]$ and $V_B, V_S : \overline{\mathbb{R}}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to [0,1]$ solve the Bellman equations (23)–(26) and satisfy the *trading condition*

$$W_B(\xi_B^1(t_B, t_S), t_B) + W_S(\xi_S^1(t_S, t_B), t_S) \leq 1 \quad \forall (t_B, t_S) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{27}$$

Remark 2. If we, instead of imposing Assumption 1, assume that upon meeting traders do not observe anything about the histories of their partners, then full trade equilibrium does not exist at least when frictions are sufficiently small. We sketch the argument as follows. Consider any full trade equilibrium in this alternative model. The steady-state finding rates $\alpha_B^\omega, \alpha_S^\omega$, the distributions of market times $F_B^\omega(\cdot), F_S^\omega(\cdot)$, and the ultimate matching probabilities m_B^ω, m_S^ω are still given by (7)–(11) and (14). But now, every buyer updates his belief only from his own market time t_B and the number of sellers n he has met. Thus, the formula (18) for his optimism level is modified as

$$\xi_B^n(t_B) = \xi_B^0 \cdot \left(\frac{\alpha_B^L}{\alpha_B^H}\right)^n \cdot \frac{\exp(-\alpha_B^L t_B)}{\exp(-\alpha_B^H t_B)}. \tag{28}$$

The beliefs $\pi_B^L(\xi_B^n(t_B))$, $\pi_B^H(\xi_B^n(t_B))$ and the search value $W_B(\xi_B^n(t_B), t_B)$ for buyers also depend on t_B and n only. Similarly, a seller's beliefs $\pi_S^L(\xi_S^n(t_S))$, $\pi_S^H(\xi_S^n(t_S))$ and search value $W_S(\xi_S^n(t_S), t_S)$ only depend on her own market time t_S and the number of buyers n she has met. For the equilibrium to have full trade, it must be the case that every buyer, when chosen to be the proposer in his first meeting, proposes the lowest price that is acceptable to every seller who has not deviated from the equilibrium; and similarly for every seller. It in particular implies that, on the equilibrium path, all buyers make the same offer (not depending on t_B) and all sellers make the same offer (not depending on t_S); hence receiving an equilibrium offer contains no information about the proposer's market time. Then, on the equilibrium path, it must be the case that every buyer, when chosen to be a proposer, offer the price $W_S(\xi_S^1(0), 0)$, since $W_S(\xi_S^1(0), 0)$ represents the outside option value of the most optimistic responder this buyer might be facing. But for the most optimistic buyer to be willing to offer $W_S(\xi_S^1(0), 0)$, it is necessary to have

$$W_B(\xi_B^1(0), 0) + W_S(\xi_S^1(0), 0) \leq 1. \tag{29}$$

Similarly, every seller's equilibrium price offer is $1 - W_B(\xi_B^1(0), 0)$; condition (29) is also a necessary condition for the most optimistic seller to be willing to offer this price. Finally, it is not hard to show that condition (29) is violated when the frictions r and δ are small. To understand this, note that $\xi_B^1(0)$ in (28) tends to ∞ as $\delta \to 0$, since $\alpha_B^L = \mu$ and $\lim_{\delta \to 0} \alpha_B^H = 0$. It follows that, when δ is small, a lucky buyer who spent virtually 0 search time to find his first partner believes that the state is L with a probability close to 1; and when r is also small his search value $W_B(\xi_B^1(0), 0)$ is close to 1. Analogously, when δ and r are small, a lucky seller who spent virtually 0 searching time to find her first partner believes that the state is H with a probability close to 1 and her search value $W_S(\xi_S^1(0), 0)$ is close to 1. A meeting between such a pair of lucky, and hence optimistic, buyer and seller cannot result in trade.[14]

3.4 Certainty benchmark

Before we tackle the aggregate uncertainty case, it is useful to investigate the full trade market equilibrium when the true state $\omega \in \{L, H\}$ is commonly known, i.e., $\phi^\omega = 1$. In this "certainty benchmark" model, the Bellman equations (23)–(26) become

$$\overline{V}_B^\omega = \beta_B \max\left\{1 - \overline{W}_S^\omega, \overline{W}_B^\omega\right\} + \beta_S \overline{W}_B^\omega, \tag{30}$$

$$\overline{V}_S^\omega = \beta_S \max\left\{1 - \overline{W}_B^\omega, \overline{W}_S^\omega\right\} + \beta_B \overline{W}_S^\omega, \tag{31}$$

$$\overline{W}_B^\omega = m_B^\omega \int e^{-rt'_B} \overline{V}_B^\omega dF_B^\omega(t'_B) = \frac{\alpha_B^\omega}{r + \delta + \alpha_B^\omega} \overline{V}_B^\omega, \tag{32}$$

$$\overline{W}_S^\omega = m_S^\omega \int e^{-rt'_S} \overline{V}_S^\omega dF_S^\omega(t'_S) = \frac{\alpha_S^\omega}{r + \delta + \alpha_S^\omega} \overline{V}_S^\omega, \tag{33}$$

where $\overline{W}_B^\omega, \overline{W}_S^\omega$ and $\overline{V}_B^\omega, \overline{V}_S^\omega$ denote the buyers' and sellers' search values and match values in state ω.

If $\overline{W}_B^\omega + \overline{W}_S^\omega \geq 1$, then (30) and (31) become $\overline{V}_B^\omega = \overline{W}_B^\omega$ and $\overline{V}_S^\omega = \overline{W}_S^\omega$, and then (32) and (33) imply $\overline{W}_B^\omega = \overline{W}_S^\omega = 0$, which is a contradiction. So any solution to the Bellman equations satisfies $\overline{W}_B^\omega + \overline{W}_S^\omega < 1$. Thus (30) and (31) become

$$\overline{V}_B^\omega = \beta_B \left(1 - \overline{W}_S^\omega\right) + \beta_S \overline{W}_B^\omega, \tag{34}$$

[14] In the model in Lauermann et al. (2018), traders do not observe any others' histories, but full trade equilibria exist when frictions are small. Our non-existence argument in Remark 2 does not apply there because there is no belief updating from searching histories in their model.

$$\overline{V}_S^\omega = \beta_S \left(1 - \overline{W}_B^\omega\right) + \beta_B \overline{W}_S^\omega. \tag{35}$$

Substituting them into (32) and (33) and solving for $\overline{W}_B^\omega, \overline{W}_S^\omega$, we obtain

$$\overline{W}_B^\omega = \frac{\beta_B \alpha_B^\omega}{r + \delta + \beta_B \alpha_B^\omega + \beta_S \alpha_S^\omega}, \tag{36}$$

$$\overline{W}_S^\omega = \frac{\beta_S \alpha_S^\omega}{r + \delta + \beta_B \alpha_B^\omega + \beta_S \alpha_S^\omega}. \tag{37}$$

Note that (36) and (37) imply $\overline{W}_B^\omega + \overline{W}_S^\omega < 1$ so that the trading condition is verified. It proves the existence and uniqueness of the full trade equilibrium under certainty.

Recall that $\alpha_B^L = \alpha_S^H = \mu$ and that α_B^H and α_S^L are at most of order δ. One can easily see that the certainty benchmark prices $\overline{p}_B^\omega = \overline{W}_S^\omega$ and $\overline{p}_S^\omega = 1 - \overline{W}_B^\omega$ converge to the Walrasian levels (i.e., 1 if $\omega = H$ and 0 if $\omega = L$) as the frictions r and δ vanish. Moreover, one can easily show that the rate of convergence is linear in $r + \delta$. Since in the steady state there are more (less) buyers than sellers in the market if the true state is H (L), the limiting prices are also Walrasian in the stock sense.

We formally state these results in the following proposition without proof.

Proposition 1 (Existence, uniqueness, and rate of convergence under certainty)**.** *Suppose the true state $\omega \in \{L, H\}$ is commonly known. Then, for every friction profile $(r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, there exists a unique full trade equilibrium. Moreover, the equilibrium search values $\overline{W}_B^\omega, \overline{W}_S^\omega$ satisfy the following properties: there exist constants $\overline{C}_0, \overline{C}_1 > 0$ not depending on r, δ such that for all $(r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$,*

$$\max \left\{\overline{W}_B^H, 1 - \overline{W}_S^H, 1 - \overline{W}_B^L, \overline{W}_S^L\right\} \leq \overline{C}_1 \cdot (r + \delta),$$

and when $r + \delta > 0$ is sufficiently small,

$$\min \left\{\overline{W}_B^H, 1 - \overline{W}_S^H, 1 - \overline{W}_B^L, \overline{W}_S^L\right\} \geq \overline{C}_0 \cdot (r + \delta).$$

That is, the discrepancy between the equilibrium transaction prices (i.e., $\overline{p}_B^L, \overline{p}_S^L$ in state L and $\overline{p}_B^H, \overline{p}_S^H$ in state H) and the Walrasian price (i.e., 0 in state L and 1 in state H) is of order $r + \delta$.

In the rest of this paper, we shall show that the results in Proposition 1 extend to the aggregate uncertainty case.

4 Uniqueness and basic equilibrium properties

From now on we tackle the aggregate uncertainty case, i.e., $\phi^\omega \in (0,1)$. Until Section 6, we shall consider *full trade equilibrium candidates*, defined as in Definition 1 except that the trading condition (27) is neglected. Of course, all the properties of the full trade equilibrium candidates (except existence) are automatically inherited by the full trade equilibria.

We first claim the existence and uniqueness of the full trade equilibrium candidate.

Proposition 2. *Under any friction profile $(r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, there exists a unique full trade equilibrium candidate.*

Proposition 2 is proved by applying the Contraction Mapping Theorem: After substituting (23) and (24) into (25) and (26), the RHS of Bellman equations (25) and (26) define a contraction on a complete metric space.

The Contraction Mapping Theorem also allows us to establish some basic properties of the full trade equilibrium candidate, which are not only interesting on their own right but also useful for proving our convergence result later.

Proposition 3 (Properties of equilibrium)**.** *In any full trade equilibrium, $W_B(\xi, t_B), W_S(\xi, t_S)$, $V_B(\xi, t_B; t_S), V_S(\xi, t_S; t_B)$ are jointly continuous in all arguments, nondecreasing in the first argument, and nonincreasing in the second argument; V_B, V_S are also nondecreasing in the third argument; moreover,*

$$\overline{V}_B^H \leq V_B(\cdot) \leq \overline{V}_B^L, \qquad \overline{V}_S^L \leq V_S(\cdot) \leq \overline{V}_S^H, \tag{38}$$

$$\overline{W}_B^H \leq W_B(\cdot) \leq \overline{W}_B^L, \qquad \overline{W}_S^L \leq W_S(\cdot) \leq \overline{W}_S^H. \tag{39}$$

(39) in particular implies $\overline{p}_B^L \leq p_B(\cdot) \leq \overline{p}_B^H$ and $\overline{p}_S^L \leq p_S(\cdot) \leq \overline{p}_S^H$; that is, relative to their certainty benchmarks, the equilibrium transaction prices include a discount in state H and a premium in state L.

The above properties are intuitive. With a fixed market time, a more optimistic trader has a higher search value. With a fixed optimism level, if an unmatched trader has spent more time in the market, his future partners after observing his market time would be more optimistic, which in turn worsens this trader's future bargaining position and reduces his current search value. Furthermore, there is an unambiguous effect of uncertainty on the equilibrium prices. In each meeting, the offered prices include an uncertainty discount if the state is H and an uncertainty premium if it is L. For example, suppose the true state is H. When traders are not certain about the state, both sides put some probability on the wrong state L where the Walrasian price is 0. This puts a downward pressure on the prices.

5 Convergence

This section states and outlines the proof of the convergence of equilibrium prices to the Walrasian ones in expectation as frictions r and δ disappear, at the rate linear in $r + \delta$. Thus, the aggregate uncertainty does not slow down the convergence relative to its certainty benchmark.

Recall that in an equilibrium-path meeting the equilibrium prices are fully characterized by the buyer's and seller's outside option values, which are their search values should the bargaining break down. So our main objective is to bound these equilibrium search values. However, there is a problem of infinite regress for bounding the equilibrium search values: These search values depend on, according to the traders' beliefs, how easily their next partners can be found, and their expected bargaining gains from future and off-equilibrium meetings with their next partners. Furthermore, any such future and off-equilibrium meeting also has its own outside option values, which are the search values of the involved traders (including the future partners of the original buyer and seller) should the bargaining in this future meeting also break down. Such search values in turn depend on, according to the involved traders' beliefs, how easily their next partners can be found, and their expected bargaining gains from further future meetings with their next partners; the involved beliefs are the original buyer's and seller's second-order beliefs concerning the state. What is even worse is that any such further future meeting also has its own outside option values, ..., and so on. Due to this problem of infinite regress, the proof of our convergence result necessarily involves off-equilibrium meetings and beliefs in unboundedly high order; moreover, although traders in any equilibrium-path meeting have a rather precise common belief most of the time, those in off-equilibrium meetings can have unboundedly different and progressively less precise (even first-order) beliefs.

To overcome the difficulty, we first bound the imprecision of a large class of off-equilibrium beliefs about the state.

To express our results succinctly, let T_B and T_S be independent (conditional on ω) random variables that follow the distributions $F_B^\omega(\cdot)$ and $F_S^\omega(\cdot)$ respectively in state ω. For

$i = 1, 2, \ldots$, let T_{Bi} and T_{Si} be independent (conditional on ω) random copies of T_B and T_S respectively. In an equilibrium-path meeting, the buyer's market time can be regarded as the random variable T_B, and the seller's market time T_S. The buyer's and seller's beliefs about state ω are common (from Lemma 2) and equal to $\pi_B^\omega(\xi_B^1(T_B, T_S)) = \pi_S^\omega(\xi_S^1(T_S, T_B))$. For our purpose it is necessary (but not sufficient) to bound the imprecision of these equilibrium-path beliefs measured as

$$\mathbb{E}\left[\pi_B^L(\xi_B^1(T_B, T_S))|\omega = H\right], \qquad \mathbb{E}\left[\pi_S^H(\xi_S^1(T_S, T_B))|\omega = L\right].$$

In fact, they are at most of order δ. The following lemma extends this result to a class of off-equilibrium beliefs that is sufficiently large for our purpose.

Lemma 3 (Belief convergence)**.** *There exist constants $c_1, c_2 > 0$ not depending on r, δ, n such that, in any full trade equilibrium under any friction profile $(r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$,*

$$\mathbb{E}\left[\max_{k\in\{1,\ldots,n\}} \pi_B^L\left(\xi_B^k\left(T_B, \sum_{i=1}^n T_{Si}\right)\right)|\omega = H\right] \leq (c_1 + c_2 n)\cdot\delta \qquad \forall n = 1, 2, \ldots, \tag{40}$$

$$\mathbb{E}\left[\max_{k\in\{1,\ldots,n\}} \pi_S^H\left(\xi_S^k\left(T_S, \sum_{i=1}^n T_{Bi}\right)\right)|\omega = L\right] \leq (c_1 + c_2 n)\cdot\delta \qquad \forall n = 1, 2, \ldots. \tag{41}$$

The class of beliefs that we bound above might look peculiar at the first glance. It is because we have made use of Lemma 2 and the monotonicity properties of beliefs and optimism levels to simplify the LHS of (40) and (41). The class of beliefs that are bounded by, say, (40) is actually larger than what it appears because, for $k_1, k_2, k_3 \leq n = 1, 2, \ldots$, Lemma 2 and the monotonicity properties imply

$$\begin{aligned}\pi_S^L\left(\xi_S^{k_3}\left(\sum_{i=1}^{k_2} T_{Si}, \sum_{i=1}^{k_1} T_{Bi}\right)\right) &\leq \pi_B^L\left(\xi_B^{k_3}\left(\sum_{i=1}^{k_1} T_{Bi}, \sum_{i=1}^{k_2} T_{Si}\right)\right)\\ &\leq \max_{k\in\{1,\ldots,n\}} \pi_B^L\left(\xi_B^k\left(T_B, \sum_{i=1}^n T_{Si}\right)\right),\end{aligned}$$

and similarly for (41). Thus, Lemma 3 implies that, given any $n = 1, 2, \ldots$, the imprecision of the belief of any trader based on at most n observations on buyers' search times and at most n observations on sellers' search times is at most of order δ, with the involved constant linearly increasing in n. (Higher n means that the beliefs occur in higher-order off-equilibrium meetings.)

Next, we show that, with $r \geq 0$ fixed, the equilibrium prices regarded as random variables converge in expectation to their true-state certainty benchmark prices as $\delta \to 0$, at the linear rate.

Proposition 4 (Convergence of prices to certainty benchmarks)**.** *There exists some constant C not depending on r, δ such that, in any full trade equilibrium under any friction profile $(r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$,*

$$\max \left\{ \begin{array}{c} \mathbb{E}\left[W_B(\xi_B^1(T_B, T_S), T_B)|\omega = H\right] - \overline{W}_B^H, \\ \overline{W}_S^H - \mathbb{E}\left[W_S(\xi_S^1(T_S, T_B), T_S)|\omega = H\right], \\ \overline{W}_B^L - \mathbb{E}\left[W_B(\xi_B^1(T_B, T_S), T_B)|\omega = L\right], \\ \mathbb{E}\left[W_S(\xi_S^1(T_S, T_B), T_S)|\omega = L\right] - \overline{W}_S^L \end{array} \right\} \leq C \cdot \delta.$$

It together with (39) implies that, with the discount rate r fixed, the expected discrepancy between the equilibrium transaction prices (i.e., $p_B(T_B, T_S)$ and $p_S(T_S, T_B)$) and the true-state certainty benchmark prices (i.e., $\overline{p}_B^L$, $\overline{p}_S^L$ in state L and $1 - \overline{p}_B^H$, $1 - \overline{p}_S^H$ in state H) is at most of order δ.

Essentially, the proof of Proposition 4 recursively and geometrically bounds the expected discrepancy between off-equilibrium search values and their no-uncertainty counterparts; more precisely, it shows that there is some $\hat{m} \in (0, 1)$ such that, for $n = 1, 2, \ldots$,

$$\begin{aligned} &\mathbb{E}\left[Q_B^n\left(T_B, \sum_{i=1}^n T_{Si}\right)|\omega = H\right] \\ &\leq \mathbb{E}\left[\Pi_B^n\left(T_B, \sum_{i=1}^n T_{Si}\right)|\omega = H\right] + \hat{m}\mathbb{E}\left[Q_B^{n+1}\left(T_B, \sum_{i=1}^{n+1} T_{Si}\right)|\omega = H\right], \end{aligned}$$

where

$$\Delta^H W_B(\cdot) \equiv W_B(\cdot) - \overline{W}_B^H, \qquad \Delta^H W_S(\cdot) \equiv \overline{W}_S^H - W_S(\cdot),$$

$$\begin{aligned} Q_B^n(t_B, t_S) \equiv \beta_B \max\left\{\Delta^H W_S(\xi_S^1(t_S, t_B), t_S), \Delta^H W_B(\xi_B^1(t_B, t_S), t_B), \Delta^H W_B(\xi_B^n(t_B, t_S), t_B)\right\} \\ + \beta_S \max\left\{\Delta^H W_B(\xi_B^1(t_B, t_S), t_B), \Delta^H W_B(\xi_B^n(t_B, t_S), t_B)\right\}, \end{aligned}$$

$$\Pi_B^n(t_B, t_S) \equiv \max\left\{\pi_B^L(\xi_B^1(t_B, t_S)), \pi_B^L(\xi_B^n(t_B, t_S))\right\}.$$

Thus, $\mathbb{E}[Q_B^1(T_B, T_S)|\omega = H]$ can be bounded by repeatedly substituting out $\mathbb{E}[Q_B^n(T_B, \sum_{i=1}^n T_{Si})|\omega = H]$ $(n = 2, 3, \ldots)$. As $\hat{m} < 1$, the resulting series with terms $\mathbb{E}[\Pi_B^n(T_B, \sum_{i=1}^n T_{Si})|\omega = H]$ $(n = 1, 2, \ldots)$ converges. It together with Lemma 3 implies $\mathbb{E}[Q_B^1(T_B, T_S)|\omega = H]$ is bounded by a constant times δ. A parallel argument works for state L and we obtain the result in Proposition 4.

Now we know, with r fixed, the equilibrium transaction prices regarded as random variables converge in expectation to the true-state certainty benchmark prices as $\delta \to 0$, and the convergence is at least as fast as δ. If both r and δ tend to 0, we know from Proposition 1 that the true-state certainty benchmark prices converge to the true-state Walrasian price at the rate linear in $r + \delta$. Thus, the equilibrium transaction prices regarded as random variables converge in expectation to the true-state Walrasian prices as $(r, \delta) \to (0, 0)$, and the convergence is at least as fast as $r + \delta$. Besides, from (39) in Proposition 3 we also know the convergence cannot be faster than that of the certainty prices, and therefore is at most as fast as $r + \delta$ too.

Corollary 1 (Convergence of prices to Walrasian levels)**.** *There exist constants $C_0, C_1 > 0$ not depending on r, δ such that, in any full trade equilibrium under any friction profile $(r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$,*

$$\max \left\{ \begin{array}{c} \mathbb{E}\left[W_B(\xi_B^1(T_B, T_S), T_B)|\omega = H\right], \\ 1 - \mathbb{E}\left[W_S(\xi_S^1(T_S, T_B), T_S)|\omega = H\right], \\ 1 - \mathbb{E}\left[W_B(\xi_B^1(T_B, T_S), T_B)|\omega = L\right], \\ \mathbb{E}\left[W_S(\xi_S^1(T_S, T_B), T_S)|\omega = L\right] \end{array} \right\} \leq C_1 \cdot (r + \delta),$$

and when $r + \delta > 0$ is sufficiently small,

$$\min \left\{ \begin{array}{c} \mathbb{E}\left[W_B(\xi_B^1(T_B, T_S), T_B)|\omega = H\right], \\ 1 - \mathbb{E}\left[W_S(\xi_S^1(T_S, T_B), T_S)|\omega = H\right], \\ 1 - \mathbb{E}\left[W_B(\xi_B^1(T_B, T_S), T_B)|\omega = L\right], \\ \mathbb{E}\left[W_S(\xi_S^1(T_S, T_B), T_S)|\omega = L\right] \end{array} \right\} \geq C_0 \cdot (r + \delta).$$

That is, the expected discrepancy between the equilibrium transaction prices (i.e., $p_B(T_B, T_S)$ and $p_S(T_S, T_B)$) and the true-state Walrasian price (i.e., 0 in state L and 1 in state H) is of order $r + \delta$.

We thus conclude that the rate of convergence of the equilibrium transaction prices is the same as it would be if the true state were commonly known.

6 Existence

This section develops the existence of the full trade equilibrium for every friction level $r+\delta > 0$. Since a unique full trade equilibrium candidate always exists (from Proposition 2), it amounts to verify the equilibrium candidate satisfies the trading condition (27).

It turns out that establishing the trading condition (27) for small $r + \delta > 0$ is highly nontrivial and it requires the exit rate δ to be sufficiently small relative to the discount rate r. To get some intuition, let us ask: why could one expect the total outside option values $W_B(\xi_B^1(t_B, t_S), t_B)+W_S(\xi_S^1(t_S, t_B), t_S)$ of equilibrium-path meetings to be uniformly bounded by 1 for all possible realizations of market times $(t_B, t_S) \in \mathbb{R}_+ \times \mathbb{R}_+$? Our idea is: First, if $\delta > 0$ is sufficiently small, in our equilibrium candidate the traders' beliefs concerning the state would be rather accurate and hence their search values would be, in a stochastic sense, close to the true-state certainty search values $\overline{W}_B^\omega, \overline{W}_S^\omega$ (as formalized by Lemma 3 and Proposition 4). Second, if the traders' beliefs were fully accurate and hence their search values exactly equal to $\overline{W}_B^\omega, \overline{W}_S^\omega$, then they would have strict incentives to trade because search is costly; indeed, from (36) and (37), the matching surplus in a meeting under certainty is

$$1 - \overline{W}_B^\omega - \overline{W}_S^\omega = \frac{r + \delta}{r + \delta + \beta_B \alpha_B^\omega + \beta_S \alpha_S^\omega}, \tag{42}$$

which is positive whenever $r + \delta > 0$. So one could hope that, when the true state is "sufficiently certain," as would be when $\delta \to 0$, the traders still have strict incentives to trade once they meet their partners.

However, such a continuity argument is problematic if we also let $r \to 0$. As made clear by (42), as $r+\delta \to 0$, the matching surplus under certainty vanishes. If the traders are only approximately indifferent between trading and not, and this is only in a stochastic sense (recalling that our convergence of search values is in expectation), we cannot guarantee the trading condition (27) to hold uniformly for all possible realizations of t_B, t_S.

In view of the above difficulty, we first let r be bounded away from 0 and establish the existence of the full trade equilibrium for all sufficiently small $\delta > 0$.

Proposition 5. *For any* $\underline{r} > 0$ *there exists a* $\bar{\bar{\delta}} > 0$ *such that, for every friction profile*

(r, δ) with $r \geq \underline{r}$ and $0 < \delta \leq \bar{\delta}$, a full trade equilibrium exists.

But since the lower bound $\underline{r} > 0$ in Proposition 5 can be made arbitrarily small, it is now easy to show that the full trade equilibrium exists under every total friction level $r + \delta > 0$.

Corollary 2 (Existence)**.** *For any $l > 0$, there exists $(r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ with $r + \delta = l$ such that a full trade equilibrium exists under the friction profile (r, δ).*

Remark 3. We conjecture that the general existence of (full trade or not) equilibria under any friction profile can be proved by a fixed point argument based on a carefully constructed self-mapping on a space of market distributions and value functions; and non-full trade equilibria exist when the exogenous exit rate is large relative to the discount rate.

7 Conclusion

We investigate a dynamic matching and bilateral bargaining market in which buyers and sellers are strategic and uncertain about the relative scarcity of the good. By studying the steady-state market equilibria with the property of full trade, we find that, as search frictions vanish, the equilibrium prices converge in expectation to the true-state Walrasian price. The rate of convergence is linear in the total search friction, which is the same as it would be if the state were commonly known.

There are many possible ways to extend our setting and analysis in the paper. We assume homogeneous traders, only two possible states, and that meeting partners observe each other's market time. We restrict attention to full trade equilibria with passive beliefs. The main purpose of these is to keep the model as tractable as possible and its departures from RW minimal. The Online Appendix extends our model by assuming meeting partners only observe imperfect signals about each other's market time, and shows that our main results are robust to this extension. What about the other possible extensions? Besides, do non-full trade equilibria exist at least when frictions are small? If so, do the transaction prices in non-full trade equilibria converge to the true-state Walrasian price? We leave these issues to future research.

Appendix

Proof of Lemma 1. Taking minimum for (4) and (5), and solving for $\min\{\Lambda_B^\omega, \Lambda_S^\omega\}$, we obtain

$$\min\{\Lambda_B^\omega, \Lambda_S^\omega\} = \frac{\min\{\lambda_B^\omega, \lambda_S^\omega\}}{\delta + \mu}.$$

Substituting back into (4) and (5), we have

$$\Lambda_B^\omega = \frac{(\delta+\mu)\lambda_B^\omega - \mu \min\{\lambda_B^\omega, \lambda_S^\omega\}}{\delta(\delta+\mu)},$$

$$\Lambda_S^\omega = \frac{(\delta+\mu)\lambda_S^\omega - \mu \min\{\lambda_B^\omega, \lambda_S^\omega\}}{\delta(\delta+\mu)}.$$

Substituting these expressions for $\Lambda_B^\omega, \Lambda_S^\omega$ into (6), we get

$$\alpha_B^\omega = \frac{\delta\mu \min\{\lambda_B^\omega, \lambda_S^\omega\}}{(\delta+\mu)\lambda_B^\omega - \mu \min\{\lambda_B^\omega, \lambda_S^\omega\}},$$

$$\alpha_S^\omega = \frac{\delta\mu \min\{\lambda_B^\omega, \lambda_S^\omega\}}{(\delta+\mu)\lambda_S^\omega - \mu \min\{\lambda_B^\omega, \lambda_S^\omega\}}.$$

These together with (1) imply (7)–(9). Solving the differential equations (2) and (3) for $F_B^\omega(\cdot)$ and $F_S^\omega(\cdot)$ with the initial conditions $F_B^\omega(0) = F_S^\omega(0) = 0$, we obtain (10) and (11). From (8),

$$\alpha_B^H = \frac{\delta\lambda_S^H}{\left(\frac{\delta}{\mu}+1\right)\lambda_B^H - \lambda_S^H} \leq \frac{\delta\lambda_S^H}{\lambda_B^H - \lambda_S^H}.$$

It proves (12). The proof of (13) is analogous. ∎

Proof of Lemma 2. From (15),

$$\xi_B^0 \xi_S^0 = \left(\frac{\lambda_B^H}{\lambda_S^H}\frac{\lambda_S^L}{\lambda_B^L}\right)^{-1}.$$

From (16), (17), and (7)–(9),

$$\xi_B^{T_B}(t_B)\xi_B^{T_S}(t_S) = \xi_B^{T_B}(0)\xi_B^{T_S}(0)\frac{\exp(-\alpha_B^L t_B - \alpha_S^L t_S)}{\exp(-\alpha_B^H t_B - \alpha_S^H t_S)},$$

$$\xi_B^{T_B}(0)\xi_B^{T_S}(0) = \frac{\alpha_B^L(\delta+\alpha_S^L)}{\alpha_B^H(\delta+\alpha_S^H)} = \frac{\lambda_S^L}{\lambda_S^H}\frac{(\delta+\mu)\lambda_B^H - \mu\lambda_S^H}{(\delta+\mu)\lambda_S^L - \mu\lambda_B^L}.$$

From (19), (20), and (7)–(9),

$$\xi_S^{T_S}(t_S)\xi_S^{T_B}(t_B) = \xi_S^{T_S}(0)\xi_S^{T_B}(0)\frac{\exp(-\alpha_B^H t_B - \alpha_S^H t_S)}{\exp(-\alpha_B^L t_B - \alpha_S^L t_S)},$$

$$\xi_S^{T_S}(0)\xi_S^{T_B}(0) = \frac{\alpha_S^H(\delta+\alpha_B^H)}{\alpha_S^L(\delta+\alpha_B^L)} = \frac{\lambda_B^H}{\lambda_B^L}\frac{(\delta+\mu)\lambda_S^L - \mu\lambda_B^L}{(\delta+\mu)\lambda_B^H - \mu\lambda_S^H}.$$

It follows that

$$\xi_B^{T_B}(t_B)\xi_B^{T_S}(t_S)\xi_S^{T_S}(t_S)\xi_S^{T_B}(t_B) = \xi_B^{T_B}(0)\xi_B^{T_S}(0)\xi_S^{T_S}(0)\xi_S^{T_B}(0) = \frac{\lambda_B^H}{\lambda_S^H}\frac{\lambda_S^L}{\lambda_B^L}$$

and hence, for $n = 1, 2, \ldots$,

$$\xi_B^n(t_B, t_S)\xi_S^n(t_S, t_B) = \left(\frac{\lambda_B^H}{\lambda_S^H}\frac{\lambda_S^L}{\lambda_B^L}\right)^{n-1}.$$

From Assumption 1,

$$\frac{\lambda_B^H}{\lambda_S^H}\frac{\lambda_S^L}{\lambda_B^L} > 1.$$

Therefore, $\xi_B^1(t_B, t_S) = 1/\xi_S^1(t_S, t_B)$ and $\xi_B^n(t_B, t_S) > 1/\xi_S^n(t_S, t_B)$ for $n = 2, 3, \ldots$ and the desired result follows. ■

Proof of Proposition 2. Let $\mathcal{B}$ denote the set of $W \equiv (W_B, W_S)$ such that both W_B and W_S are functions from $\overline{\mathbb{R}}_+ \times \mathbb{R}_+$ to $[0, 1]$. Define a mapping $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ as follows. For any $W \in \mathcal{B}$, let $V \equiv (V_B, V_S)$, defined on $\overline{\mathbb{R}}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, be given by (23) and (24); then let $\mathcal{T}(W) \equiv (\mathcal{T}_B(W), \mathcal{T}_S(W))$, defined on $\overline{\mathbb{R}}_+ \times \mathbb{R}_+$, be given by the RHS of (25) and (26). The range of such $\mathcal{T}(W)$ is contained in $[0, 1]$, so that $\mathcal{T}(W) \in \mathcal{B}$. In the following we claim that $\mathcal{T}$ is a contraction on the nonempty complete metric space $(\mathcal{B}, d_\infty)$, where d_∞ denotes the supremum metric (or uniform metric).

Consider any $W, \hat{W} \in \mathcal{B}$ and let $\hat{d} \equiv d_\infty(W, \hat{W})$. Since for any $x_1, x_2, x_3, x_4 \in \mathbb{R}$,[15]

$$|\max\{x_1, x_2\} - \max\{x_3, x_4\}| \leq \max\{|x_1 - x_3|, |x_2 - x_4|\},$$

$V_B, \hat{V}_B, V_S, \hat{V}_S$ given by (23) and (24) satisfy

$$d_\infty(V_B, \hat{V}_B) \leq \beta_B \max\left\{d_\infty(W_S, \hat{W}_S), d_\infty(W_B, \hat{W}_B)\right\} + \beta_S d_\infty(W_B, \hat{W}_B) \leq \hat{d},$$

$$d_\infty(V_S, \hat{V}_S) \leq \beta_S \max\left\{d_\infty(W_B, \hat{W}_B), d_\infty(W_S, \hat{W}_S)\right\} + \beta_B d_\infty(W_S, \hat{W}_S) \leq \hat{d}.$$

Then $\mathcal{T}_B(W), \mathcal{T}_B(\hat{W})$ given by (25) satisfy, for any $(\xi, t) \in \overline{\mathbb{R}}_+ \times \mathbb{R}_+$,

$$\begin{aligned}(\mathcal{T}_B(W) - \mathcal{T}_B(\hat{W}))(\xi, t) &\leq \sum_\omega \pi_B^\omega(\xi) m_B^\omega \int e^{-rt'_B} \hat{d} dF_B^\omega(t'_B) \\ &= \sum_\omega \pi_B^\omega(\xi) \frac{\alpha_B^\omega}{r + \delta + \alpha_B^\omega} \hat{d} \\ &\leq \max\left\{\frac{\alpha_B^L}{r + \delta + \alpha_B^L}, \frac{\alpha_B^H}{r + \delta + \alpha_B^H}\right\} \hat{d} \\ &= \frac{\mu}{r + \delta + \mu} \cdot \hat{d}.\end{aligned}$$

Let $k \equiv \frac{\mu}{r+\delta+\mu} < 1$. Then $d_\infty(\mathcal{T}_B(W), \mathcal{T}_B(\hat{W})) \leq k\hat{d}$. One can similarly prove $d_\infty(\mathcal{T}_S(W), \mathcal{T}_S(\hat{W})) \leq k\hat{d}$ so that $d_\infty(\mathcal{T}(W), \mathcal{T}(\hat{W})) \leq k\hat{d}$. We conclude that $\mathcal{T}$ is a contraction on nonempty complete metric space $(\mathcal{B}, d_\infty)$. The Contraction Mapping Theorem implies that $\mathcal{T}$ has a unique fixed point; this fixed point corresponds to our unique full trade equilibrium candidate. ■

Proof of Proposition 3. Here we follow the notations and results established in the proof of Proposition 2. In that proof, we have seen that the self-map $\mathcal{T}$ is a contraction on the nonempty complete metric space $(\mathcal{B}, d_\infty)$, where d_∞ denotes the supremum metric, and the unique full trade equilibrium candidate can be identified with the unique fixed point of $\mathcal{T}$. Let $\mathcal{B}'$ denote the set of $W \equiv (W_B, W_S) \in \mathcal{B}$ that satisfies the claimed properties about W_B, W_S. Note that $\mathcal{B}'$ is a closed set in the metric space $(\mathcal{B}, d_\infty)$.

[15]This follows from the following chain of inequalities, assuming without loss of generality $x_1 = \max\{x_1, x_2, x_3, x_4\}$:

$$|\max\{x_1, x_2\} - \max\{x_3, x_4\}| = |x_1 - \max\{x_3, x_4\}| = x_1 - \max\{x_3, x_4\} \leq x_1 - x_3 \leq \max\{|x_1 - x_3|, |x_2 - x_4|\}.$$

We next claim that, for any $W \in \mathcal{B}'$, we have $\mathcal{T}(W) \in \mathcal{B}'$. So fix any $W \in \mathcal{B}'$. The continuity and monotonicity properties of W_B, W_S and $\xi_B^{T_B}, \xi_B^{T_S}, \xi_S^{T_S}, \xi_S^{T_B}, \xi_B^1, \xi_S^1$ imply that $V_B(\xi, t_B; t_S), V_S(\xi, t_S; t_B)$ given by (23) and (24) are jointly continuous in all arguments, nondecreasing in the first and third arguments, and nonincreasing in the second argument; moreover, (23), (24), and (39) imply

$$V_B(\cdot) \geq \beta_B \left(1 - \overline{W}_S^H\right) + \beta_S \overline{W}_B^H = \overline{V}_B^H,$$

$$V_B(\cdot) \leq \beta_B \max\left\{1 - \overline{W}_S^L, \overline{W}_B^L\right\} + \beta_S \overline{W}_B^L = \overline{V}_B^L,$$

$$V_S(\cdot) \geq \beta_S \left(1 - \overline{W}_B^L\right) + \beta_B \overline{W}_S^L = \overline{V}_S^L,$$

$$V_S(\cdot) \leq \beta_S \max\left\{1 - \overline{W}_B^H, \overline{W}_S^H\right\} + \beta_B \overline{W}_S^H = \overline{V}_S^H.$$

Now consider $\mathcal{T}_B(W)$ given by V_B and the RHS of (25). We claim that $\mathcal{T}_B(W)(\xi, t_B)$ is continuous in (ξ, t_B), nondecreasing in ξ, and nonincreasing in t_B. The continuity follows from the continuity of V_B (and all other involved functions in the RHS of (25)). An increase in t_B makes the value of V_B in the integrand in (25) weakly smaller. Thus, $\mathcal{T}_B(W)(\xi, t_B)$ is nonincreasing in t_B. Now consider an increasing in ξ. The increase in ξ makes the double integral in (25) weakly larger, for both $\omega = L$ and $\omega = H$. Since $\alpha_B^H < \alpha_B^L$ (and hence $F_B^H(\cdot)$ first-order stochastically dominates $F_B^L(\cdot)$) and $\alpha_S^L < \alpha_S^H$ (and hence $F_S^L(\cdot)$ first-order stochastically dominates $F_S^H(\cdot)$), the integral in (25) is weakly larger when evaluated at $\omega = L$ than when evaluated at $\omega = H$, because t'_B is smaller and t'_S larger when $\omega = L$ in the sense of stochastic dominance, which in turn makes the value of V_B and $e^{-rt'_B}$ weakly larger in expectation. These together with $m_B^L > m_S^H$ imply that the product of m_B^ω and the integral is weakly larger when evaluated at $\omega = L$ than when evaluated at $\omega = H$. Recalling that $\pi_B^L(\xi)$ is strictly increasing in ξ, we obtain that $\mathcal{T}_B(W)(\xi, t_B)$ is nondecreasing in ξ. Similarly, $\mathcal{T}_S(W)(\xi, t_S)$, given by V_S and the RHS of (26), is continuous in (ξ, t_S), nondecreasing in ξ, and nonincreasing in t_S. Furthermore, the above results also imply

$$\mathcal{T}_B(W)(\cdot) \geq m_B^H \int e^{-rt'_B} \overline{V}_B^H dF_B^H(t'_B) = \overline{W}_B^H,$$

$$\mathcal{T}_B(W)(\cdot) \leq m_B^L \int e^{-rt'_B} \overline{V}_B^L dF_B^L(t'_B) = \overline{W}_B^L,$$

$$\mathcal{T}_S(W)(\cdot) \geq m_S^L \int e^{-rt'_S} \overline{V}_S^L dF_S^L(t'_S) = \overline{W}_S^L,$$

$$\mathcal{T}_S(W)(\cdot) \leq m_S^H \int e^{-rt'_S} \overline{V}_S^H dF_S^H(t'_S) = \overline{W}_S^H.$$

The above arguments show that $\mathcal{T}(\mathcal{B}') \subseteq \mathcal{B}'$. Since $\mathcal{T}$ is a contraction and $\mathcal{B}'$ is a closed set, the unique fixed point of $\mathcal{T}$ (which corresponds to the unique full trade equilibrium candidate) belongs to $\mathcal{B}'$. ■

Proof of Lemma 3. We provide the proof of (40) only. Using (18) and (7), we have, for $k = 1, \ldots, n$,

$$\xi_B^k \left(T_B, \sum_{i=1}^n T_{Si} \right) = \xi_B^0 \left(\frac{\mu}{\alpha_B^H} \frac{\delta + \alpha_S^L}{\delta + \mu} \right)^k e^{-(\mu - \alpha_B^H) T_B} e^{(\mu - \alpha_S^L)(T_{S1} + \cdots + T_{Sn})}. \quad (43)$$

Let

$$a \equiv \frac{\lambda_B^H}{\lambda_S^H} \frac{\lambda_S^L}{\lambda_S^L - \lambda_B^L}.$$

(1) implies $a > 1$. Using (8) and (13),

$$\frac{\mu}{\alpha_B^H} \frac{\delta + \alpha_S^L}{\delta + \mu} \leq \frac{\mu}{\alpha_B^H(\delta + \mu)} \left(\delta + \frac{\delta \lambda_B^L}{\lambda_S^L - \lambda_B^L} \right) = \frac{(\delta + \mu)\lambda_B^H - \mu \lambda_S^H}{(\delta + \mu)\lambda_S^H} \frac{\lambda_S^L}{\lambda_S^L - \lambda_B^L} \leq \frac{\lambda_B^H}{\lambda_S^H} \frac{\lambda_S^L}{\lambda_S^L - \lambda_B^L} = a.$$

It follows that

$$\left(\frac{\mu}{\alpha_B^H} \frac{\delta + \alpha_S^L}{\delta + \mu} \right)^k \leq a^n$$

no matter whether the above LHS is greater than or smaller than 1. Then (43) implies

$$\xi_B^k \left(T_B, \sum_{i=1}^n T_{Si} \right) \leq \xi_B^0 a^n e^{-(\mu - \alpha_B^H) T_B} e^{(\mu - \alpha_S^L)(T_{S1} + \cdots + T_{Sn})}.$$

Recall that, in state H, the density of T_B is $(\delta + \alpha_B^H) e^{-(\delta + \alpha_B^H) t_B}$. Thus, for any $t_S \geq 0$,

$$\mathbb{E} \left[\max_{k \in \{1, \ldots, n\}} \pi_B^L \left(\xi_B^k \left(T_B, \sum_{i=1}^n T_{Si} \right) \right) | T_{S1} + \cdots + T_{Sn} = t_S, \omega = H \right]$$
$$\leq \int_0^\infty \frac{\xi_B^0 a^n e^{-(\mu - \alpha_B^H) t_B} e^{(\mu - \alpha_S^L) t_S}}{1 + \xi_B^0 a^n e^{-(\mu - \alpha_B^H) t_B} e^{(\mu - \alpha_S^L) t_S}} (\delta + \alpha_B^H) e^{-(\delta + \alpha_B^H) t_B} dt_B. \quad (44)$$

The last line can be rewritten as

$$\frac{\delta+\alpha_B^H}{\delta+\mu}\int_0^\infty \frac{\xi_B^0 a^n(\delta+\mu)e^{-(\delta+\mu)t_B}e^{(\mu-\alpha_S^L)t_S}}{1+\xi_B^0 a^n e^{-(\delta+\mu)t_B}e^{(\delta+\alpha_B^H)t_B}e^{(\mu-\alpha_S^L)t_S}}dt_B. \tag{45}$$

Using the fact that $e^{(\delta+\alpha_B^H)t_B}\geq 1$, the integral in (45) is bounded from above by

$$\begin{aligned}
&\int_0^\infty \frac{\xi_B^0 a^n(\delta+\mu)e^{-(\delta+\mu)t_B}e^{(\mu-\alpha_S^L)t_S}}{1+\xi_B^0 a^n e^{-(\delta+\mu)t_B}e^{(\mu-\alpha_S^L)t_S}}dt_B \\
&= -\int_0^\infty \frac{d}{dt_B}\ln\left(1+\xi_B^0 a^n e^{-(\delta+\mu)t_B}e^{(\mu-\alpha_S^L)t_S}\right)dt_B \\
&= \ln\left(1+\xi_B^0 a^n e^{(\mu-\alpha_S^L)t_S}\right) \\
&\leq \ln(1+\xi_B^0)+n\ln a+\mu t_S.
\end{aligned} \tag{46}$$

Therefore, the conditional expected value in (44) is bounded from above by

$$\frac{\delta+\alpha_B^H}{\delta+\mu}\left(\ln(1+\xi_B^0)+n\ln a+\mu t_S\right),$$

which is linear in t_S. Replacing t_S with the random variable $T_{S1}+\cdots+T_{Sn}$ and taking expectation conditional on $\omega=H$, we have

$$\mathbb{E}\left[\max_{k\in\{1,\ldots,n\}}\pi_B^L\left(\xi_B^k\left(T_B,\sum_{i=1}^n T_{Si}\right)\right)|\omega=H\right]\leq\frac{\delta+\alpha_B^H}{\delta+\mu}\left(\ln(1+\xi_B^0)+n\ln a+n\right)$$

since $\mathbb{E}[T_{S1}+\cdots+T_{Sn}|\omega=H]=n\mathbb{E}[T_S|\omega=H]=\frac{n}{\delta+\mu}\leq\frac{n}{\mu}$. It together with (12) implies

$$\begin{aligned}
&\mathbb{E}\left[\max_{k\in\{1,\ldots,n\}}\pi_B^L\left(\xi_B^k\left(T_B,\sum_{i=1}^n T_{Si}\right)\right)|\omega=H\right] \\
&\leq\frac{\delta+\alpha_B^H}{\mu}\left(\ln(1+\xi_B^0)+(1+\ln a)n\right) \\
&\leq\frac{1}{\mu}\left(\delta+\frac{\delta\lambda_S^H}{\lambda_B^H-\lambda_S^H}\right)\left(\ln(1+\xi_B^0)+(1+\ln a)n\right) \\
&=\delta\frac{\lambda_B^H}{\mu(\lambda_B^H-\lambda_S^H)}\left[\ln\left(1+\frac{\phi^L\lambda_B^L}{\phi^H\lambda_B^H}\right)+\left(1+\ln\left(\frac{\lambda_B^H}{\lambda_S^H}\frac{\lambda_S^L}{\lambda_S^L-\lambda_B^L}\right)\right)n\right].
\end{aligned}$$

Therefore, (40) holds with

$$c_1 \equiv \frac{\lambda_B^H}{\mu(\lambda_B^H - \lambda_S^H)} \ln\left(1 + \frac{\phi^L \lambda_B^L}{\phi^H \lambda_B^H}\right) > 0,$$

$$c_2 \equiv \frac{\lambda_B^H}{\mu(\lambda_B^H - \lambda_S^H)} \left[1 + \ln\left(\frac{\lambda_B^H}{\lambda_S^H} \frac{\lambda_S^L}{\lambda_S^L - \lambda_B^L}\right)\right] > 0.$$

■

Proof of Proposition 4. We only provide a proof for $\omega = H$; the proof for $\omega = L$ is parallel.

We first introduce some shorthand. Let

$$\Delta^H W_B(\cdot) \equiv W_B(\cdot) - \overline{W}_B^H, \qquad \Delta^H W_S(\cdot) \equiv \overline{W}_S^H - W_S(\cdot),$$

$$\Delta^H V_B(\cdot) \equiv V_B(\cdot) - \overline{V}_B^H, \qquad \Delta^H V_S(\cdot) \equiv \overline{V}_S^H - V_S(\cdot).$$

From Proposition 3 these are all nonnegative; their monotonicity properties are also inherited from those of $W_B(\cdot), W_S(\cdot), V_B(\cdot), V_S(\cdot)$. For $n = 1, 2, \dots$ and $t_B, t_S \geq 0$, define

$$\begin{aligned} Q_B^n(t_B, t_S) \equiv \beta_B \max \left\{\Delta^H W_S(\xi_S^1(t_S, t_B), t_S), \Delta^H W_B(\xi_B^1(t_B, t_S), t_B), \Delta^H W_B(\xi_B^n(t_B, t_S), t_B)\right\} \\ + \beta_S \max \left\{\Delta^H W_B(\xi_B^1(t_B, t_S), t_B), \Delta^H W_B(\xi_B^n(t_B, t_S), t_B)\right\}, \end{aligned}$$

$$\Pi_B^n(t_B, t_S) \equiv \max \left\{\pi_B^L(\xi_B^1(t_B, t_S)), \pi_B^L(\xi_B^n(t_B, t_S))\right\}.$$

From (18) $\xi_B^n(\cdot)$ is monotone (either nondecreasing or nonincreasing depending on $\alpha_B^L, \alpha_B^H, \alpha_S^L, \alpha_S^H$) in n. It follows that $Q_B^n(\cdot)$ and $\Pi_B^n(\cdot)$ are nondecreasing in n. Indeed, if $\xi_B^n(\cdot)$ is nondecreasing in n, the claim is clear; if $\xi_B^n(\cdot)$ is nonincreasing in n, then $\xi_B^1(t_B, t_S) \geq \xi_B^n(t_B, t_S)$ for all $n = 1, 2, \dots$ and $t_B, t_S \geq 0$, so that $Q_B^n(\cdot)$ and $\Pi_B^n(\cdot)$ do not depend on n.

Subtracting (32) with $\omega = H$ from (25) with $t_B = 0$, we get

$$\begin{aligned} \Delta^H W_B(\xi, 0) &= \sum_\omega \pi_B^\omega(\xi) \left(m_B^\omega \mathbb{E}\left[e^{-rT_B} V_B(\xi\xi_B^{T_B}(T_B), T_B; T_S)|\omega\right] - m_B^H \mathbb{E}\left[e^{-rT_B} \overline{V}_B^H |H\right]\right) \\ &\leq \pi_B^L(\xi) + m_B^H \mathbb{E}\left[\Delta^H V_B(\xi\xi_B^{T_B}(T_B), T_B; T_S)|H\right]. \end{aligned}$$

Subtracting (34) with $\omega = H$ from (23), we get

$$\begin{aligned}
&\Delta^H V_B(\xi\xi_B^{T_B}(T_B), T_B; T_S) \\
&\leq \beta_B \max\left\{\Delta^H W_S(\xi_S^1(T_S, T_B), T_S), \Delta^H W_B(\xi\xi_B^{T_B}(T_B)\xi_B^{T_S}(T_S), T_B)\right\} \\
&\quad + \beta_S \max\left\{\Delta^H W_B(\xi_B^1(T_B, T_S), T_B), \Delta^H W_B(\xi\xi_B^{T_B}(T_B)\xi_B^{T_S}(T_S), T_B)\right\}.
\end{aligned}$$

Taking $\xi = \xi_B^n(t_B, t_S)$ and using the monotonicity properties of $\Delta^H W_B$, $\Delta^H W_S$, ξ_B^1, ξ_S^1, ξ_B^{n+1} and the fact that $\xi_B^n(t_B, t_S)\xi_B^{T_B}(T_B)\xi_B^{T_S}(T_S) = \xi_B^{n+1}(t_B + T_B, t_S + T_S)$, we get

$$\begin{aligned}
&\Delta^H V_B(\xi_B^n(t_B, t_S)\xi_B^{T_B}(T_B), T_B; T_S) \\
&\leq \beta_B \max\left\{\Delta^H W_S(\xi_S^1(T_S, T_B), T_S), \Delta^H W_B(\xi_B^{n+1}(t_B + T_B, t_S + T_S), T_B)\right\} \\
&\quad + \beta_S \max\left\{\Delta^H W_B(\xi_B^1(T_B, T_S), T_B), \Delta^H W_B(\xi_B^{n+1}(t_B + T_B, t_S + T_S), T_B)\right\} \\
&\leq \beta_B \max\left\{\Delta^H W_S(\xi_S^1(t_S + T_S, T_B), t_S + T_S), \Delta^H W_B(\xi_B^{n+1}(T_B, t_S + T_S), T_B)\right\} \\
&\quad + \beta_S \max\left\{\Delta^H W_B(\xi_B^1(T_B, t_S + T_S), T_B), \Delta^H W_B(\xi_B^{n+1}(T_B, t_S + T_S), T_B)\right\} \\
&\leq Q_B^{n+1}(T_B, t_S + T_S).
\end{aligned}$$

The above results imply, for any $t_B, t_S \geq 0$,

$$\begin{aligned}
\Delta^H W_B(\xi_B^n(t_B, t_S), t_B) &\leq \Delta^H W_B(\xi_B^n(t_B, t_S), 0) \\
&\leq \pi_B^L(\xi_B^n(t_B, t_S)) + m_B^H \mathbb{E}\left[\Delta^H V_B(\xi_B^n(t_B, t_S)\xi_B^{T_B}(T_B), T_B; T_S)|H\right] \\
&\leq \Pi_B^n(t_B, t_S) + m_B^H \mathbb{E}\left[Q_B^{n+1}(T_B, t_S + T_S)|H\right]. \qquad (47)
\end{aligned}$$

In particular, when $n = 1$,

$$\Delta^H W_B(\xi_B^1(t_B, t_S), t_B) \leq \Pi_B^1(t_B, t_S) + m_B^H \mathbb{E}\left[Q_B^2(T_B, t_S + T_S)|H\right]. \qquad (48)$$

Subtracting (26) from (33) with $\omega = H$, we get

$$\begin{aligned}
\Delta^H W_S(\xi, t_S) &= \sum_\omega \pi_S^\omega(\xi)\left(m_S^H \mathbb{E}\left[e^{-rT_S}\overline{V}_S^H|H\right] - m_S^\omega \mathbb{E}\left[e^{-rT_S} V_S(\xi\xi_S^{T_S}(T_S), t_S + T_S; T_B)|\omega\right]\right) \\
&\leq \pi_S^L(\xi) + \mathbb{E}\left[\Delta^H V_S(\xi\xi_S^{T_S}(T_S), t_S + T_S; T_B)|H\right].
\end{aligned}$$

Subtracting (24) from (35) with $\omega = H$, we get

$$\begin{aligned}
&\Delta^H V_S(\xi\xi_S^{T_S}(T_S), t_S + T_S; T_B) \\
&\leq \beta_S \Delta^H W_B(\xi_B^1(T_B, t_S + T_S), T_B) + \beta_B \Delta^H W_S(\xi_S^1(t_S + T_S, T_B), t_S + T_S) \\
&\leq Q_B^1(T_B, t_S + T_S).
\end{aligned}$$

Combining the above two inequalities, taking $\xi = \xi_S^1(t_S, t_B)$, and using $\pi_S^L(\xi_S^1(t_S, t_B)) = \pi_B^L(\xi_B^1(t_B, t_S))$ from Lemma 2, we get

$$\begin{aligned}
\Delta^H W_S(\xi_S^1(t_S, t_B), t_S) &\leq \pi_B^L(\xi_S^1(t_S, t_B)) + \mathbb{E}\left[Q_B^1(T_B, t_S + T_S)|H\right] \\
&\leq \Pi_B^1(t_B, t_S) + \mathbb{E}\left[Q_B^1(T_B, t_S + T_S)|H\right]. \qquad (49)
\end{aligned}$$

(47), (48), and (49) imply

$$\begin{aligned}
&\max\left\{\Delta^H W_S(\xi_S^1(t_S, t_B), t_S), \Delta^H W_B(\xi_B^1(t_B, t_S), t_B), \Delta^H W_B(\xi_B^n(t_B, t_S), t_B)\right\} \\
&\leq \Pi_B^n(t_B, t_S) + \mathbb{E}\left[Q_B^{n+1}(T_B, t_S + T_S)|H\right], \qquad (50)
\end{aligned}$$

where we have used that $Q_B^n(\cdot)$ and $\Pi_B^n(\cdot)$ are nondecreasing in n. Similarly, (47) and (48) imply

$$\begin{aligned}
&\max\left\{\Delta^H W_B(\xi_B^1(t_B, t_S), t_B), \Delta^H W_B(\xi_B^n(t_B, t_S), t_B)\right\} \\
&\leq \Pi_B^n(t_B, t_S) + m_B^H \mathbb{E}\left[Q_B^{n+1}(T_B, t_S + T_S)|H\right]. \qquad (51)
\end{aligned}$$

Multiplying (50) by β_B and (51) by β_S, and summing the inequalities, we obtain

$$Q_B^n(t_B, t_S) \leq \Pi_B^n(t_B, t_S) + (\beta_B + \beta_S m_B^H)\mathbb{E}\left[Q_B^{n+1}(T_B, t_S + T_S)|H\right]. \qquad (52)$$

Note that

$$m_B^H = \frac{\mu \lambda_S^H}{(\delta + \mu)\lambda_B^H} \leq \frac{\lambda_S^H}{\lambda_B^H} < 1.$$

Let

$$\hat{m}_B \equiv \beta_B + \beta_S \frac{\lambda_S^H}{\lambda_B^H} < 1$$

so that

$$\beta_B + \beta_S m_B^H \le \hat{m}_B < 1. \tag{53}$$

Taking $t_S = \sum_{i=1}^n t_{Si}$ in (52), multiplying both sides by $f_B^H(t_B)$ and $f_S^H(t_{S1}) \cdots f_S^H(t_{Sn})$, taking integral with respect to $(t_B, t_{S1}, \ldots, t_{Sn})$ over $[0, \infty)^{n+1}$, and using (53), we obtain, for $n = 1, 2, \ldots$,

$$\hat{Q}_n \le \hat{\Pi}_n + \hat{m}\hat{Q}_{n+1} \tag{54}$$

where

$$\hat{Q}_n \equiv \mathbb{E}\left[Q_B^n\left(T_B, \sum_{i=1}^n T_{Si}\right)|H\right],$$

$$\hat{\Pi}_n \equiv \mathbb{E}\left[\Pi_B^n\left(T_B, \sum_{i=1}^n T_{Si}\right)|H\right].$$

From Lemma 3, we know there exist constants $c_1, c_2 > 0$ not depending on r, δ, n such that, for $n = 1, 2, \ldots$,

$$\hat{\Pi}_n \le (c_1 + c_2 n)\delta.$$

Hence, from (48) and (49), we have

$$\begin{aligned}
&\max\left\{\mathbb{E}\left[\Delta^H W_B(\xi_B^1(T_B, T_S), T_B)|H\right], \mathbb{E}\left[\Delta^H W_S(\xi_S^1(T_S, T_B), T_S)|H\right]\right\} \\
&\le \hat{\Pi}_1 + \hat{Q}_2 \\
&\le (c_1 + c_2)\delta + \hat{Q}_2.
\end{aligned}$$

From (54), we have

$$\hat{Q}_2 \le \sum_{n=2}^{\infty} \hat{m}^{n-2}\hat{\Pi}_n \le \delta \sum_{n=2}^{\infty} \hat{m}^{n-2}(c_1 + c_2 n). \tag{55}$$

It is well known that $0 < \hat{m} < 1$ implies the series $\sum_n \hat{m}^n$ and $\sum_n n\hat{m}^n$ are convergent[16] so that the series in (55) is convergent. It proves that there exists a constant C, not depending on r, δ, such that

$$\max\left\{\mathbb{E}\left[\Delta^H W_B(\xi_B^1(T_B, T_S), T_B)|H\right], \mathbb{E}\left[\Delta^H W_S(\xi_S^1(T_S, T_B), T_S)|H\right]\right\} \le C \cdot \delta$$

[16] Indeed, $\sum_{n=1}^{\infty} \hat{m}^n = \hat{m}/(1-\hat{m})$ and $\sum_{n=1}^{\infty} n\hat{m}^n = \hat{m}/(1-\hat{m})^2$.

as desired. ■

Proof of Corollary 1. It follows from Proposition 1, (39) in Proposition 3, and Proposition 4. ■

The proof of Proposition 5 requires the following lemma, which reduces the domain of condition (27) to the cases of $t_B = 0$ and of $t_S = 0$.

Lemma 4. *Under every friction profile, the full trade equilibrium candidate satisfies (27) if and only if it satisfies (27) on* $\{(t_B, t_S) \in \mathbb{R}_+^2 : t_B t_S = 0\}$.

Proof of Lemma 4. Fix a friction profile. Since $\alpha_B^L = \mu > \alpha_B^H$ and $\alpha_S^H = \mu > \alpha_S^L$, every point in $\mathbb{R}_+^2$ can be regarded as some $(t_B + \Delta t_B, t_S + \Delta t_S)$ with $t_B, t_S, \Delta t_B, \Delta t_S \in \mathbb{R}_+$, $t_B t_S = 0$, and $(\alpha_B^L - \alpha_B^H)\Delta t_B = (\alpha_S^H - \alpha_S^L)\Delta t_S$. Consider any such $(t_B + \Delta t_B, t_S + \Delta t_S)$. From (18) and (21) we know that $(\alpha_B^L - \alpha_B^H)\Delta t_B = (\alpha_S^H - \alpha_S^L)\Delta t_S$ implies $\xi_B^1(t_B + \Delta t_B, t_S + \Delta t_S) = \xi_B^1(t_B, t_S)$ and $\xi_S^1(t_S + \Delta t_S, t_B + \Delta t_B) = \xi_B^1(t_S, t_B)$. It together with the monotonicity properties in Proposition 3 implies that

$$W_B(\xi_B^1(t_B + \Delta t_B, t_S + \Delta t_S), t_B + \Delta t_B) = W_B(\xi_B^1(t_B, t_S), t_B + \Delta t_B) \leq W_B(\xi_B^1(t_B, t_S), t_B),$$

$$W_S(\xi_S^1(t_S + \Delta t_S, t_B + \Delta t_B), t_S + \Delta t_S) = W_S(\xi_S^1(t_S, t_B), t_S + \Delta t_S) \leq W_S(\xi_S^1(t_S, t_B), t_S).$$

Therefore, (27) is satisfied at $(t_B + \Delta t_B, t_S + \Delta t_S)$ whenever it is satisfied at (t_B, t_S). ■

Proof of Proposition 5. Fix any $\underline{r} > 0$. From Lemma 4, it suffices to show, for all $r \geq \underline{r}$ and sufficiently small $\delta > 0$, first, $W_B(\xi_B^1(0, t_S), 0) + W_S(\xi_S^1(t_S, 0), t_S) < 1$ for all $t_S \in \mathbb{R}_+$, and second, $W_B(\xi_B^1(t_B, 0), t_B) + W_S(\xi_S^1(0, t_B), 0) < 1$ for all $t_B \in \mathbb{R}_+$. In the following we prove the first claim; the second claim can be proved by parallel argument.

We use the random variable notations in Section 5. Substituting (23), (24) into (25), (26) and taking $t_B = 0$ and $\xi = \xi_B^1(0, t_S), \xi_S^1(t_S, 0)$, we have

$$W_B(\xi_B^1(0, t_S), 0) = \sum_{\omega} \pi_B^{\omega}(\xi_B^1(0, t_S)) m_B^{\omega} \times$$
$$\mathbb{E}\left[e^{-rT_B} \begin{pmatrix} \beta_B \max\{1 - W_S(\xi_S^1(T_S, T_B), T_S), W_B(\xi_B^2(T_B, t_S + T_S), T_B)\} \\ +\beta_S \max\{W_B(\xi_B^1(T_B, T_S), T_B), W_B(\xi_B^2(T_B, t_S + T_S), T_B)\} \end{pmatrix} |\omega \right], \quad (56)$$

$$W_S(\xi_S^1(t_S,0),t_S) = \sum_\omega \pi_S^\omega(\xi_S^1(t_S,0))m_S^\omega \times$$
$$\mathbb{E}\left[e^{-rT_S}\begin{pmatrix} \beta_S \max\{1-W_B(\xi_B^1(T_B,t_S+T_S),T_B), W_S(\xi_S^2(t_S+T_S,T_B),t_S+T_S)\} \\ +\beta_B \max\{W_S(\xi_S^1(t_S+T_S,T_B),t_S+T_S), W_S(\xi_S^2(t_S+T_S,T_B),t_S+T_S)\}\end{pmatrix}|\omega\right]. \tag{57}$$

Keep in mind that W_B and W_S also implicitly depend on (r,δ). In the following Steps 1–2, we shall prove that, for any $\bar{t}\geq 0$ and $\bar{r}\geq \underline{r}$,

$$\sup_{(t_S,r)\in[0,\bar{t}]\times[\underline{r},\bar{r}]}\{W_B(\xi_B^1(0,t_S),0)+W_S(\xi_S^1(t_S,0),t_S)\}<1 \tag{58}$$

for all sufficiently small $\delta>0$. In Step 3 we shall take care of the case of unbounded t_S. (The case of unbounded r is trivial.)

Step 1. Fix any $\bar{t}\geq 0$. The proof of Lemma 3 can be modified to show that, given any $t_S\in[0,\bar{t}]$, for $j=B,S$, $n\geq 1$, and $\omega\neq\omega'$,

$$\mathbb{E}\left[\max_{k\in\{1,\dots,n\}}\pi_B^L\left(\xi_B^k\left(T_B,t_S+\sum_{i=1}^n T_{Si}\right)\right)|\omega=H\right]\leq (c_1+c_2 n)\cdot\delta \qquad \forall n=1,2,\dots, \tag{59}$$

$$\mathbb{E}\left[\max_{k\in\{1,\dots,n\}}\pi_S^H\left(\xi_S^k\left(t_S+T_S,\sum_{i=1}^n T_{Bi}\right)\right)|\omega=L\right]\leq (c_1+c_2 n)\cdot\delta \qquad \forall n=1,2,\dots. \tag{60}$$

for some constants $c_1,c_2>0$ not depending on r,δ,n,t_S. Indeed, (60) is implied by (41) since the LHS of (60) decreases with t_S. The proof of (59) is the same as that of (40) except that the role of $\sum_{i=1}^n T_{Si}$ is replaced by $t_S+\sum_{i=1}^n T_{Si}$, so that $\mathbb{E}[T_{S1}+\cdots+T_{Sn}|\omega=H]\leq\frac{n}{\mu}$ becomes $\mathbb{E}[t_S+T_{S1}+\cdots+T_{Sn}|\omega=H]\leq\bar{t}+\frac{n}{\mu}$ and the constant c_1 becomes

$$c_1\equiv\frac{\lambda_B^H}{\mu(\lambda_B^H-\lambda_S^H)}\left(\ln\left(1+\frac{\phi^L}{\phi^H}\frac{\lambda_B^L}{\lambda_B^H}\right)+\mu\bar{t}\right).$$

Step 2. We claim that, for any $\bar{t}\geq 0$ and $\bar{r}\geq\underline{r}$, (58) holds for all sufficiently small $\delta>0$.

Fix any $\bar{t}\geq 0$ and $\bar{r}\geq\underline{r}$. Using Step 1, the proof of Proposition 4 can be modified to show that, given any $t_S \in [0, \bar{t}]$ and $n = 1, 2, \ldots$,

$$\max \left\{ \begin{array}{c} \mathbb{E}\left[W_B(\xi_B^n(T_B, t_S + T_S), T_B)|\omega = H\right] - \overline{W}_B^H, \\ \overline{W}_S^H - \mathbb{E}\left[W_S(\xi_S^n(t_S + T_S, T_B), t_S + T_S)|\omega = H\right], \\ \overline{W}_B^L - \mathbb{E}\left[W_B(\xi_B^n(T_B, t_S + T_S), T_B)|\omega = L\right], \\ \mathbb{E}\left[W_S(\xi_S^n(t_S + T_S, T_B), t_S + T_S)|\omega = L\right] - \overline{W}_S^L \end{array} \right\} \leq C \cdot \delta, \tag{61}$$

where C is some constant not depending on r, δ, n, t_S.

Since $[0, \bar{t}] \times [\underline{r}, \bar{r}]$ is a compact set and the bound $C \cdot \delta$ converges monotonically to 0, by Dini's Theorem, we can conclude that, as $\delta \to 0$, the four elements inside the $\max\{\cdot\}$ in the LHS of (61), regarded as functions of (t_S, r) on $[0, \bar{t}] \times [\underline{r}, \bar{r}]$, all converge to 0 uniformly. Then, from (56) and (57), we have

$$\begin{aligned} &\limsup_{\delta \to 0} \sup_{(t_S, r) \in [0, \bar{t}] \times [\underline{r}, \bar{r}]} W_B(\xi_B^1(0, t_S), 0) \\ &\leq \limsup_{\delta \to 0} \sup_{(t_S, r) \in [0, \bar{t}] \times [\underline{r}, \bar{r}]} \sum_{\omega} \pi_B^{\omega}(\xi_B^1(0, t_S)) m_B^{\omega} \mathbb{E}\left[e^{-rT_B}|\omega\right] \left(\beta_B(1 - \overline{W}_S^{\omega}) + \beta_S \overline{W}_B^{\omega}\right) \\ &= \limsup_{\delta \to 0} \sup_{(t_S, r) \in [0, \bar{t}] \times [\underline{r}, \bar{r}]} \sum_{\omega} \pi_B^{\omega}(\xi_B^1(0, t_S)) \overline{W}_B^{\omega} \end{aligned} \tag{62}$$

and

$$\begin{aligned} &\limsup_{\delta \to 0} \sup_{(t_S, r) \in [0, \bar{t}] \times [\underline{r}, \bar{r}]} W_S(\xi_S^1(t_S, 0), t_S) \\ &\leq \limsup_{\delta \to 0} \sup_{(t_S, r) \in [0, \bar{t}] \times [\underline{r}, \bar{r}]} \sum_{\omega} \pi_S^{\omega}(\xi_S^1(t_S, 0)) m_S^{\omega} \mathbb{E}\left[e^{-rT_S}|\omega\right] \left(\beta_S(1 - \overline{W}_B^{\omega}) + \beta_S \overline{W}_S^{\omega}\right) \\ &= \limsup_{\delta \to 0} \sup_{(t_S, r) \in [0, \bar{t}] \times [\underline{r}, \bar{r}]} \sum_{\omega} \pi_S^{\omega}(\xi_S^1(t_S, 0)) \overline{W}_S^{\omega}. \end{aligned} \tag{63}$$

Adding the inequalities in (62) and (63), and recalling $\pi_B^{\omega}(\xi_B^1(0, t_S)) = \pi_S^{\omega}(\xi_S^1(t_S, 0))$ from

Lemma 2, we have

$$\begin{aligned}
&\limsup_{\delta\to 0} \sup_{(t_S,r)\in[0,\bar t]\times[\underline{r},\bar r]} \left(W_B(\xi_B^1(0,t_S),0)+W_S(\xi_S^1(t_S,0),t_S)\right)\\
&\le \limsup_{\delta\to 0} \sup_{(t_S,r)\in[0,\bar t]\times[\underline{r},\bar r]} \sum_\omega \pi_B^\omega(\xi_B^1(0,t_S))(\overline{W}_B^\omega+\overline{W}_S^\omega)\\
&\le \limsup_{\delta\to 0} \sup_{r\in[\underline{r},\bar r]} \max\left\{\overline{W}_B^L+\overline{W}_S^L, \overline{W}_B^H+\overline{W}_S^H\right\}\\
&= \limsup_{\delta\to 0} \sup_{r\in[\underline{r},\bar r]} \max\left\{\frac{\beta_B\alpha_B^L+\beta_S\alpha_S^L}{r+\delta+\beta_B\alpha_B^L+\beta_S\alpha_S^L}, \frac{\beta_B\alpha_B^H+\beta_S\alpha_S^H}{r+\delta+\beta_B\alpha_B^H+\beta_S\alpha_S^H}\right\}\\
&= \max\left\{\frac{\beta_B\mu}{\underline{r}+\beta_B\mu}, \frac{\beta_S\mu}{\underline{r}+\beta_S\mu}\right\}\\
&< 1,
\end{aligned}$$

where the second last line is from (7)–(9). It completes the proof of (58) for all sufficiently small $\delta > 0$.

Since the $\bar t$ and $\bar r$ in (58) can be arbitrarily large, it now remains to show the limit of $W_B(\xi_B^1(0,t_S),0)+W_S(\xi_S^1(t_S,0),t_S)$ as $t_S\to\infty$ or $r\to\infty$ (or both) is smaller than 1 for all sufficiently small $\delta>0$. But it is easy to see that $W_B(\xi_B^1(0,t_S),0)+W_S(\xi_S^1(t_S,0),t_S)\to 0$ as $r\to\infty$. So it remains to consider the limit as $t_S\to\infty$. The next step tackles this last case.

Step 3. We claim that, for any $\bar r \ge \underline{r}$,

$$\limsup_{\delta\to 0, t_S\to\infty} \sup_{r\in[\underline{r},\bar r]} [W_B(\xi_B^1(0,t_S),0)+W_S(\xi_S^1(t_S,0),t_S)] < 1.$$

From Proposition 3, we know $W_S(\xi_S^1(t_S,0),t_S)$ is nonincreasing in t_S, so that

$$\limsup_{\delta\to 0, t_S\to\infty} \sup_{r\in[\underline{r},\bar r]} W_S(\xi_S^1(t_S,0),t_S) \le \limsup_{\delta\to 0} \sup_{(t_S,r)\in[0,\bar t]\times[\underline{r},\bar r]} W_S(\xi_S^1(t_S,0),t_S) \tag{64}$$

for any $\bar t \ge 0$. Using (63), the RHS of (64) is not greater than

$$\limsup_{\delta\to 0} \sup_{(t_S,r)\in[0,\bar t]\times[\underline{r},\bar r]} \sum_\omega \pi_S^\omega(\xi_S^1(t_S,0))\overline{W}_S^\omega. \tag{65}$$

Since $\overline{W}_S^\omega$ is decreasing in r, $\pi_S^H(\xi_S^1(t_S,0))$ is decreasing in t_S, and $\overline{W}_S^L < \overline{W}_S^H$, (65) can be written as

$$\limsup_{\delta\to 0} \sum_{\omega} \pi_S^{\omega}(\xi_S^1(\bar{t},0))\overline{W}_S^{\omega}|_{r=\underline{r}}.$$

Since (64) is true for any $\bar{t} \geq 0$, we have

$$\limsup_{\delta\to 0, t_S\to\infty} \sup_{r\in[\underline{r},\bar{r}]} W_S(\xi_S^1(t_S,0),t_S) \leq \inf_{\bar{t}\geq 0} \limsup_{\delta\to 0} \sum_{\omega} \pi_S^{\omega}(\xi_S^1(\bar{t},0))\overline{W}_S^{\omega}|_{r=\underline{r}}. \tag{66}$$

Notice that, for any $\bar{\delta} > 0$,

$$\lim_{t_S\to\infty} \sup_{\delta\in(0,\bar{\delta}]} \pi_S^H(\xi_S^1(t_S,0)) = 0.$$

Therefore,

$$\inf_{\bar{t}\geq 0} \limsup_{\delta\to 0} \pi_S^H(\xi_S^1(t_S,0)) = 0$$

and hence

$$\inf_{\bar{t}\geq 0} \limsup_{\delta\to 0} \sum_{\omega} \pi_S^{\omega}(\xi_S^1(\bar{t},0))\overline{W}_S^{\omega}|_{r=\underline{r}} = \overline{W}_S^L|_{(r,\delta)=(\underline{r},0)}.$$

Thus, (66) becomes

$$\limsup_{\delta\to 0, t_S\to\infty} \sup_{r\in[\underline{r},\bar{r}]} W_S(\xi_S^1(t_S,0),t_S) \leq \overline{W}_S^L|_{(r,\delta)=(\underline{r},0)}.$$

Recalling that $W_S(\cdot) \geq \overline{W}_S^L$ and $W_B(\cdot) \leq \overline{W}_B^L$ from Proposition 3, we have

$$\limsup_{\delta\to 0, t_S\to\infty} \sup_{r\in[\underline{r},\bar{r}]} W_S(\xi_S^1(t_S,0),t_S) = \overline{W}_S^L|_{(r,\delta)=(\underline{r},0)}$$

and

$$\begin{aligned}
&\limsup_{\delta\to 0, t_S\to\infty} \sup_{r\in[\underline{r},\bar{r}]} [W_B(\xi_B^1(0,t_S),0) + W_S(\xi_S^1(t_S,0),t_S)] \\
&\leq \left(\limsup_{\delta\to 0, t_S\to\infty} \sup_{r\in[\underline{r},\bar{r}]} W_B(\xi_B^1(0,t_S),0)\right) + \left(\limsup_{\delta\to 0, t_S\to\infty} \sup_{r\in[\underline{r},\bar{r}]} W_S(\xi_S^1(t_S,0),t_S)\right) \\
&\leq \overline{W}_B^L|_{(r,\delta)=(\underline{r},0)} + \overline{W}_S^L|_{(r,\delta)=(\underline{r},0)} \\
&= \frac{\beta_B\mu}{\underline{r}+\beta_B\mu} < 1,
\end{aligned}$$

as desired. ∎

Proof of Corollary 2. Fix any $l > 0$. Pick some $\underline{r} \in (0, l)$. From Proposition 5 there exists some $\bar{\delta} > 0$ such that a full trade equilibrium exists under every friction profile (r, δ) with $r \geq \underline{r}$ and $0 < \delta \leq \bar{\delta}$. Now, if $\underline{r} + \bar{\delta} \geq l$, then we can set $r = \underline{r}$ and $\delta = l - \underline{r} \in (0, \bar{\delta}]$ so that a full trade equilibrium exists under (r, δ). If $\underline{r} + \bar{\delta} < l$, then we can set $r = l - \bar{\delta} > \underline{r}$ and $\delta = \bar{\delta}$ so that a full trade equilibrium exists under (r, δ). ∎

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