Bilateral Matching and Bargaining with Private Information

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Abstract

We study equilibria of a dynamic matching and bargaining game (DMBG) with two-sided private information bilateral bargaining. The model is a private information replica of Mortensen and Wright (2002). There are two kinds of frictions: time discounting and explicit search costs. A simple necessary and sufficient condition on parameters for existence of a non-trivial equilibrium is obtained. This condition is the same regardless whether the information is private or not. In addition, it is shown that when the discount rate is sufficiently small, the equilibrium is unique and has the property that every meeting results in trade.

Keywords: Matching and Bargaining, Search Frictions, Two-sided Incomplete Information, Diamond's paradox

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1 Introduction

The famous Diamond paradox (Diamond (1971)) implies a complete breakdown of a market with costly search in which all the bargaining power is given to one party, even when the search costs are small. But in many markets, both sides have at least some bargaining power. In this paper, we explore the interaction between asymmetric bargaining power, costly search and private information in a dynamic matching and bargaining game (DMBG). We establish an upper efficiency bound for all equilibria of our model. This bound implies that market efficiency can be very low if the bargaining power is sufficiently asymmetric,

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even when frictions are small.[1] We prove a simple necessary and sufficient condition for (no) market breakdown. Furthermore, we show that when the discount rate is small, the equilibrium is unique and involves full trade.

Our model is a private information replica of Mortensen and Wright (2002; MW hereafter).[2] Buyers and sellers arrive continuously to the market, engage in costly search, are matched pairwise, and bargain under private information. The bargaining protocol is random-proposal: the buyer makes a final offer with probability $\alpha_B > 0$ and the seller makes a final offer with the complementary probability $\alpha_S > 0$. The parameters α_B, α_S are interpreted as the buyers' and sellers' relative bargaining power. Search frictions are parameterized by a discount rate $r > 0$ and explicit search costs incurred at rates κ_B for buyers and κ_S for sellers. The buyers and sellers are heterogeneous in their types (the valuation of the good for a buyer, or the cost of providing the good for a seller), which are private information to them.

If one side of the market, say the buyers, is given most of the bargaining power, i.e. α_B is close to 1, then a softer version of the Diamond paradox arises. On the other side of the market, there is a marginal type of seller who makes zero economic profit and is indifferent between searching or not. Since search is costly and this seller has only tiny bargaining power, this seller can break even only when she has a very high finding rate. With a constant returns to scale matching technology, this high rate must be supported by a high market ratio of buyers to sellers. But a high ratio of buyers to sellers in turn implies that the finding rate for buyers is very low, so only the most avid ones, if any, will choose to engage in costly search. The volume of the trading activity is likely to be very low, and a complete market breakdown is a possibility.

MW have shown this effect under complete information bargaining, and in a class of *full trade equilibria*, in which every meeting results in trade. Such equilibria sometimes (but not always) exist. In this paper, we develop a general theory of equilibria in DMBG with private information bargaining and completely characterize (no) market breakdown.

In our model, a buyer's finding rate $\ell_B(\zeta)$ is decreasing in the market tightness ζ, the ratio of the (stock) masses of buyers and sellers in the market. The seller's finding rate $\ell_S(\zeta)$ is increasing in ζ. Whether or not the market is viable is determined by the incentives to engage in costly search by some traders from both sides. We show that it occurs if and only if there is some ζ such that $\ell_B(\zeta)\alpha_B > \kappa_B$ and $\ell_S(\zeta)\alpha_S > \kappa_S$. This condition is easy to check because, as we show, one only need to check it at the full trade equilibrium market tightness $\zeta_0 = \alpha_B\kappa_S/(\alpha_S\kappa_B)$.

The intuition for our condition is as follows. In order to avoid market breakdown, positive masses of traders from both sides should participate. Hence the most avid buyers (with valuation 1) and the most avid sellers (with cost 0) should strictly prefer to enter. A meeting of such a pair of traders yields the gains from trade equal to 1 (due to our normalization). Suppose for a moment that the bargaining is under complete information so that the gains from trade are simply divided according to the distribution of bargaining

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[1] Hosios (1990) showed that search markets are generally inefficient.

[2] It is also similar to Shneyerov and Wong (2008) who assume that time is in discrete periods, as in Satterthwaite and Shneyerov (2007), while MW assume continuous time. MW model is an extension of Rubinstein and Wolinsky (1985) and Gale (1986), Gale (1987) to general bilateral matching technologies and continuous type distributions.

power (α_B, α_S). Then it is clear that the most avid traders from both sides would strictly prefer to enter if both $\ell_B(\zeta)\alpha_B > \kappa_B$ and $\ell_S(\zeta)\alpha_S > \kappa_S$ are satisfied. We do not know what the market tightness ζ would be, but in equilibrium it is adjusted so that both inequalities hold, provided that such a ζ exists. Of course we assume private information bargaining so that the division of gains from trade is not simply determined by (α_B, α_S), but as we show it does not affect our existence condition.

The no market breakdown condition has an equivalent form that allows further interpretation. The expected cost of search incurred by a buyer-seller pair until the next meeting, $K(\zeta)$, depends on ζ through the expected waiting times of the buyers and sellers. We show that the maximal surplus of a buyer-seller pair in the market is bounded from above by $1 - K(\zeta_0)$, and the market is viable iff $K(\zeta_0) < 1$. Even when search costs are small, the market can break down if the bargaining power is sufficiently asymmetric: if, ceteris paribus, say α_B moves closer to 0, then the upper bound $1 - K(\zeta_0)$ moves closer to 0, and a complete market breakdown always occurs for some small $\alpha_B > 0$.

As in Satterthwaite and Shneyerov (2007) (with incomplete information) and MW (with complete information), equilibria exist in which all matches result in trade. These are full trade equilibria. We show that, if the discount rate r is sufficiently small relative to the search cost, the equilibrium is unique and is full-trade. The intuition is the "homogenizing" effect of making r small: the traders become progressively more homogeneous in their responding decisions. All active buyers will reject offers that are more than an epsilon over the reservation price of the marginal buyer. This provides extremely strong incentives for sellers to propose that price, which will be accepted by all buyers. Similar logic applies to the other side of the market.

The literature on DMBG originated with Rubinstein and Wolinsky (1985, 1990), Gale (1986), Gale (1987) and is by now voluminous. Most of this literature, e.g. Butters (1979), Rubinstein and Wolinsky (1985, 1990), Wolinsky (1988), De Fraja and Sakovics (2001), Serrano (2002), Moreno and Wooders (2002), Lauermann (2006a), Lauermann (2006b), Satterthwaite and Shneyerov (2008), Lauermann (2008), did not consider costly search, a prerequisite for Diamond-type effects that arise in our model. The problem of market viability that we study doesn't arise there. As a matter of fact, Lauermann (2008) shows convergence to perfect competition when one side of the market is given all the bargaining power. In that literature, showing existence of a non-trivial equilibrium can only be mathematically hard.

The literature on costly search, including Satterthwaite and Shneyerov (2007), Shneyerov and Wong (2008) and Atakan (2008a,b), has focussed on small frictions and convergence to perfect competition.[3] Gale (1987) assumes symmetric bargaining power and does not address the existence of a non-trivial equilibrium. Satterthwaite and Shneyerov (2007) consider first-price auctions. Shneyerov and Wong (2008) study convergence to perfect competition of general matching and bargaining mechanisms, including the mechanism considered here. For *fixed* α_B and α_S, Shneyerov and Wong (2008) show that, as the frictions $(\kappa_B, \kappa_S, r) \to 0$, all equilibria converge to perfect competition, and do so at the optimal rate. But our analysis in this paper implies that even for small fixed frictions, all equilibria will be highly inefficient if the bargaining power is sufficiently asymmetric. The

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[3]Dynamic matching models have also been used to study markets for durable goods, as in Inderst and Müller (2002), or to study screening, as in Inderst (2001).

Diamond effect can severely restrict efficiency of the market even with small frictions.

Atakan (2009) also considers a DMBG with costly search, with a focus on convergence to perfect competition.[4] He proves a general existence result assuming Free First Draw (FD): the sellers with cost below some small ε must enter for at least one period. As in our method of proof, this assumption forces entry into the market. But we prove that traders will in fact enter voluntarily iff $K(\zeta_0) < 1$.

We are not aware of a paper that shows existence of a non-trivial equilibrium in a model with costly search in such a general way as here. In particular, no paper that we know of goes all the way to showing that a non-trivial equilibrium exists *if and only if* search costs are below an explicit threshold, which here is shown to be the same under full and private information. Shneyerov and Wong (2008) consider a model with auctions, and only prove a local existence result in the class of full trade equilibria. MW consider a full information model and prove a global existence result for full trade equilibria. However, even when full trade equilibria do not exist, non full trade equilibria may exist.[5] Also, the proofs of existence of full trade equilibrium in these papers only use relatively elementary methods such as the Implicit Function Theorem. In contrast, the proof of our general existence result relies on a functional fixed point theorem applied to a carefully constructed self-map of a compact, convex subset of a Banach space.

The structure of the paper is as follows. Section 2 introduces our model. Section 3 states the general existence theorem and gives the main ingredients of its proof. Section 4 explores full trade equilibria. The proofs we do not provide in the text are in the Appendix.

2   The Model

The agents in our model are potential buyers and sellers of a homogeneous, indivisible good. Each buyer has a unit demand for the good, while each seller has unit supply. All traders are risk neutral. Potential buyers are heterogeneous in their valuations (or types) v of the good. Potential sellers are also heterogeneous in their costs (or types) c of providing the good. For simplicity, we assume $v, c \in [0, 1]$. Time is continuous and infinite horizon. The instantaneous discount rate is $r > 0$. The details of the model are as follows:

- Entry: Potential buyers and sellers are continuously born at rates b and s respectively. The types of new-born buyers are drawn i.i.d. from the c.d.f. $F(v)$ and the types of new-born sellers are drawn i.i.d. from the c.d.f. $G(c)$. Each trader's type will not change once it is drawn. Entry (or participation, or being active) is voluntary. Potential traders decide whether to enter the market once they are born. Those who do not enter will get zero payoff. Those who enter must incur the search cost continuously at the rate $\kappa_B > 0$ for buyers and $\kappa_S > 0$ for sellers, until they leave the market.

- Matching: Active buyers and sellers are randomly and continuously matched pairwise

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[4]Atakan (2008) considers a related model, but with no discounting (so that all equilibria are full-trade), and characterizes the set of equilibria as the set of solutions to a hypothetical social planner's problem.

[5]This is shown in the 2008 working paper version of this paper, available at `http://artyom239.googlepages.com/shneyerov_and_wong_bilateral_oct2008.pdf`.

with the instantaneous rate of matching given by a matching function $M(B,S)$, where B and S are the masses of active buyers and active sellers currently in the market.

- Bargaining: Once a pair of buyer and seller is matched, they bargain without observing the type of their partner. The bargaining protocol is *random-proposal*: with probability $\alpha_S \in (0,1)$, the seller makes a take-it-or-leave-it offer to the buyer, then the buyer chooses either to accept or reject. And with probability $\alpha_B = 1 - \alpha_S$, the buyer proposes and the seller responds. We also assume the market is anonymous, so that the bargainers do not know their partners' market history, e.g. how long they have been in the market, what they proposed previously, and what offers they rejected previously.

- If a type v buyer and a type c seller trade at a price p, then they leave the market with payoff $v - p$, and $p - c$ respectively. If bargaining between the matched pair breaks down, both traders can either stay in the market waiting for another match as if they were never matched, or simply exit and never come back.

We make the following assumptions on the primitives of our model.

Assumption 1 (distributions of inflow types) *The cumulative distributions $F(v)$ and $G(c)$ of inflow types have densities $f(v)$ and $g(c)$ on $(0,1)$, bounded away from 0 and ∞: $0 < \underline{f} \le f(v) \le \bar{f} < \infty$, $0 < \underline{g} \le g(c) \le \bar{g} < \infty$.*

Assumption 2 (matching function) *The matching function M is continuous on $\mathbb{R}_+^2$, nondecreasing in each argument, exhibits constant returns to scale (i.e. is homogeneous of degree one), and satisfies $M(0,S) = M(B,0) = 0$.*

We will restrict attention to steady states. Let $\zeta \equiv B/S$ be a (steady-state) ratio of buyers to sellers, and define $m(\zeta) \equiv M(\zeta, 1)$. Note that $m(\zeta)$ and $m(\zeta)/\zeta$ are nondecreasing and nonincreasing respectively in ζ, and m is continuous on $\mathbb{R}_{++}$. In this notation, the Poisson arrival rates for buyers and sellers become $\ell_B(\zeta) \equiv m(\zeta)/\zeta$ and $\ell_S(\zeta) \equiv m(\zeta)$.

Our definition of a non-trivial search equilibrium parallels that in MW. Let $W_B, W_S : [0,1] \to \mathbb{R}_+$ be the search value functions for buyers and sellers, and let $N_B, N_S : [0,1] \to \mathbb{R}_+$ be the (stock) market mass functions, i.e. $N_B(v)$ is the mass of buyers in the market with valuations less than v and $N_S(c)$ is the mass of sellers with costs less than c. (In this notation, $B = N_B(1)$ and $S = N_S(1)$.) Also let $\chi_B, \chi_S : [0,1] \to \{0,1\}$ be the entry strategies, i.e. a buyer with valuation v enters iff $\chi_B(v) = 1$. A new element of equilibrium in our model with incomplete information is the proposing strategies $p_B, p_S : [0,1] \to [0,1]$.

Sequential optimality requires that the values of search in steady state satisfy the following Bellman equations. For a type v buyer,

$$rW_B(v) = \max_{\chi \in \{0,1\}} \chi \cdot \{\ell_B(\zeta)[\alpha_B \pi_B(v) + \alpha_S \int\limits_{\{c:v-p_S(c) \ge W_B(v)\}} (v - p_S(c) - W_B(v))\frac{dN_S(c)}{S}] - \kappa_B\}$$

(1)

where $\pi_B(v)$ is the buyer's capital gain when he becomes a proposer:

$$\pi_B(v) = \max_{p \in [0,1]} \int\limits_{\{c:p-c \ge W_S(c)\}} (v - p - W_B(v))\frac{dN_S(c)}{S}.$$

(2)

The buyer's equilibrium entry strategy $\chi_B(v)$ must be an optimal value of χ in (1), and his equilibrium proposing strategy $p_B(v)$ must be an optimal value of p in (2). The intuition for (1) is that, contingent on entry, a buyer's flow value of search $rW_B(v)$ is equal to the expected capital gain due to matching a partner, net of the flow search cost. Specifically, the buyer's proposed price $p_B(v)$ is accepted by the seller if her trade surplus is weakly greater than the value of search, i.e. if $p_B(v) - c \geq W_S(c)$. The seller's proposed price $p_S(c)$ is accepted by the buyer if his trade surplus is weakly greater than his value of search, i.e. if $v - p_S(c) \geq W_B(v)$. When the buyer trades, the capital gain is his trade surplus minus the value of search. The Bellman equation for the sellers has a similar form:

$$rW_S(c) = \max_{\chi \in \{0,1\}} \chi \cdot \{\ell_S(\zeta)[\alpha_S \pi_S(c) + \alpha_B \int_{\{v: p_B(v) - c \geq W_S(c)\}} (p_B(v) - c - W_S(c)) \frac{dN_B(v)}{B}] - \kappa_S\} \tag{3}$$

where

$$\pi_S(c) = \max_{p \in [0,1]} \int_{\{v: v - p \geq W_B(v)\}} (p - c - W_S(c)) \frac{dN_B(v)}{B}. \tag{4}$$

An optimal value of χ in (3) is the seller's equilibrium entry strategy $\chi_S(c)$, and an optimal value of p in (4) is her equilibrium proposing strategy $p_S(c)$.[6]

It is convenient to define the trading probabilities in a given meeting, $q_B(v)$ for buyers and $q_S(c)$ for sellers:

$$q_B(v) = \alpha_B \int_{\{c: p_B(v) - c \geq W_S(c)\}} \frac{dN_S(c)}{S} + \alpha_S \int_{\{c: v - p_S(c) \geq W_B(v)\}} \frac{dN_S(c)}{S},$$

$$q_S(c) = \alpha_S \int_{\{v: v - p_S(c) \geq W_B(v)\}} \frac{dN_B(v)}{B} + \alpha_B \int_{\{v: p_B(v) - c \geq W_S(c)\}} \frac{dN_B(v)}{B}.$$

In steady state, the rate of inflow of the traders of each type is equal to the rate of the outflow due to trading:

$$b\chi_B(v) dF(v) = \ell_B(\zeta) q_B(v) dN_B(v), \tag{5}$$

$$s\chi_S(c) dG(c) = \ell_S(\zeta) q_S(c) dN_S(c). \tag{6}$$

Definition 1 *Parallel to MW, a non-trivial steady-state search equilibrium, or a non-trivial equilibrium, is defined as a tuple $(W_B, W_S, \chi_B, \chi_S, p_B, p_S, N_B, N_S)$ such that $B \equiv N_B(1) > 0$, $S \equiv N_S(1) > 0$, equations (1), (3), (5) and (6) hold, and χ_B, p_B, χ_S, p_S solve the optimization problems in (1), (2), (3) and (4) respectively.*

The following lemma shows that in equilibrium traders use cutoff entry strategies, and characterizes the value functions W_B and W_S.

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[6]In MW, traders observe their partners' types, and fully extract their rents; the proposing strategies are $p_B(c) = c + W_S(c)$ and $p_S(v) = v - W_B(v)$.

Lemma 1 *In any nontrivial steady-state equilibrium, there are marginal entering types $\underline{v}, \bar{c} \in (0,1)$ such that the supports of N_B and N_S are $[\underline{v}, 1]$ and $[0, \bar{c}]$ respectively. Marginal entrants (i.e. type $\underline{v}$ buyers and type $\bar{c}$ sellers) are indifferent between entering or not, while the entry preferences of all others are strict. $\{v : \chi_B(v) = 1\}$ is either $[\underline{v}, 1]$ or $(\underline{v}, 1]$. $\{c : \chi_S(c) = 1\}$ is either $[0, \bar{c}]$ or $[0, \bar{c})$. W_B is absolutely continuous, convex, strictly increasing on $[\underline{v}, 1]$, with $W_B(\underline{v}) = 0$; whenever differentiable, $W_B'(v) = \chi_B(v) \ell_B q_B(v) / [r + \ell_B q_B(v)]$. W_S is absolutely continuous, convex, strictly decreasing on $[0, \bar{c}]$, with $W_S(\bar{c}) = 0$; whenever differentiable, $W_S'(c) = -\chi_S(c) \ell_S q_S(c) / [r + \ell_S q_S(c)]$. The trading probability q_B is strictly positive and nondecreasing on $[\underline{v}, 1]$, while q_S is strictly positive and nonincreasing on $[0, \bar{c}]$.*

Proof. In the Appendix. ■

We call the $\underline{v}$ and $\bar{c}$ in Lemma 1 the *marginal buyers' type* and *marginal sellers' type*. Since the flow and stock masses of marginal buyers and marginal sellers (who are indifferent between entering or not) are zero, we without loss of generality assume they enter, i.e. $\chi_B(\underline{v}) = \chi_S(\bar{c}) = 1$. The following lemma characterizes proposing strategies and gives the indifference conditions for the marginal traders.

Lemma 2 *In any non-trivial steady-state equilibrium, the proposing strategies $p_B(v)$ and $p_S(c)$ are nondecreasing on $[\underline{v}, 1]$ and $[0, \bar{c}]$ respectively. Moreover, for all $v \in [\underline{v}, 1]$, $p_B(v) < v - W_B(v)$ and $p_B(v) \in [W_S(0), \bar{c}]$; for all $c \in [0, \bar{c}]$, $p_S(c) > c + W_S(c)$ and $p_S(c) \in [\underline{v}, 1 - W_B(1)]$. Furthermore, $\ell_B(\zeta) \alpha_B \pi_B(\underline{v}) = \kappa_B$ and $\ell_S(\zeta) \alpha_S \pi_S(\bar{c}) = \kappa_S$.*

Proof. In the Appendix. ■

The intuition for Lemma 2 is as follows. Consider e.g. buyers. We show that their proposing strategies are always below what they are willing to accept, i.e. $p_B(v) < v - W_B(v)$. Also, they never choose to waste and propose more than acceptable to the marginal seller, i.e. $p_B(v) \leq \bar{c}$, and they never propose a price so low that it is unacceptable even to the seller with $c = 0$, i.e. $p_B(v) \geq W_S(0)$. Also, the marginal buyers only make profit from bargaining when they propose. Since $W_B(\underline{v}) = 0$, they are willing to accept $\underline{v}$ and in equilibrium this is commonly known to the sellers. Because $\underline{v}$ is the lowest type of buyer active in the market, the sellers would be wasting money proposing less than $\underline{v}$. Therefore $\ell_B(\zeta) \alpha_B \pi_B(\underline{v}) = \kappa_B$.

3   A simple necessary and sufficient condition for no market breakdown

Since search is costly, there is a possibility that the market completely breaks down (i.e. a non-trivial equilibrium does not exist) because no trader finds it worthwhile to enter the market. We show that whether the market is viable is determined by a simple condition.

Theorem 1 (No market breakdown) *A non-trivial steady-state equilibrium exists if and only if $K\left(\zeta_0\right) < 1$, where*

$$K\left(\zeta\right) = \frac{\kappa_B}{\ell_B\left(\zeta\right)} + \frac{\kappa_S}{\ell_S\left(\zeta\right)},$$
$$\zeta_0 = \left(\alpha_B\kappa_S\right)/\left(\alpha_S\kappa_B\right).$$

The function $K\left(\zeta\right)$ is the expected search costs incurred by a pair of buyer and seller when the market tightness is ζ and there is no discounting. This condition is best understood in an equivalent form given in the following lemma, which also derives a min-max representation of $K\left(\zeta_0\right)$ that will be used in the sequel.

Lemma 3 *For any matching function satisfying Assumption 2, the following statements are equivalent. (i) $K\left(\zeta_0\right) < 1$. (ii) For some $\zeta > 0$, we have $\ell_B\left(\zeta\right)\alpha_B > \kappa_B$ and $\ell_S\left(\zeta\right)\alpha_S > \kappa_S$.*

Proof. Observe that $\kappa_B/\left(\ell_B\left(\zeta\right)\alpha_B\right)$ is nondecreasing and $\kappa_S/\left(\ell_S\left(\zeta\right)\alpha_S\right)$ nonincreasing in ζ; and they are equal if and only if $\zeta = \zeta_0$. Then it is easy to see that

$$K\left(\zeta_0\right) = \min_{\zeta>0}\max\left\{\frac{\kappa_B}{\ell_B\left(\zeta\right)\alpha_B}, \frac{\kappa_S}{\ell_S\left(\zeta\right)\alpha_S}\right\} = \max_{\zeta>0}\min\left\{\frac{\kappa_B}{\ell_B\left(\zeta\right)\alpha_B}, \frac{\kappa_S}{\ell_S\left(\zeta\right)\alpha_S}\right\}. \quad (7)$$

The statement in the lemma follows from the first equality. ∎

If $\ell_B\left(\zeta\right)\alpha_B > \kappa_B$ and $\ell_S\left(\zeta\right)\alpha_S > \kappa_S$, then the most efficient traders, the buyers with valuations around 1 and sellers with costs around 0, have a mutual incentive to engage in search if the other side does so. Given that the trade surplus when they meet is 1, they will be able to recover their search costs when they propose. This effectively "jump starts" the market. The actual market tightness ζ will adjust as long as there is some ζ satisfying $\ell_B\left(\zeta\right)\alpha_B > \kappa_B$ and $\ell_S\left(\zeta\right)\alpha_S > \kappa_S$.

Remark 1 *Theorem 1 implies that Diamond paradox also holds here: if one side of the market has almost all the bargaining power, say $\alpha_B \rightarrow 0$, then $\ell_B\left(\zeta\right)\alpha_B > \kappa_B$ could hold only when $\zeta \rightarrow 0$, but then $\ell_S\left(\zeta\right)\alpha_S > \kappa_S$ would be violated since $\ell_S\left(\zeta\right) \rightarrow 0$; thus the market must break down.*

The necessity of the condition $K\left(\zeta_0\right) < 1$ is established by the following proposition that also derives an upper efficiency bound in any equilibrium.

Proposition 1 *In any non-trivial steady-state equilibrium, we have*

$$0 < W_B\left(1\right) + W_S\left(0\right) < 1 - K\left(\zeta_0\right).$$

Proof. From Lemma 1, $W_B\left(1\right) > 0$ and $W_S\left(0\right) > 0$, proving the first inequality. The marginal type equations in Lemma 2 can be written as

$$\ell_B\left(\zeta\right)\alpha_B\left[\underline{v} - p_B\left(\underline{v}\right)\right]\int_{\{c:p_B(\underline{v})-c\geq W_S(c)\}}\frac{dN_S(c)}{S} = \kappa_B, \quad (8)$$

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$$\ell_S(\zeta)\alpha_S \left[p_S\left(\bar{c}\right) - \bar{c}\right] \int_{\{v:v-p_S(\bar{c})\geq W_B(v)\}} \frac{dN_B(v)}{B} = \kappa_S. \tag{9}$$

The equation (8) implies

$$\ell_B\left(\zeta\right)\alpha_B\left[1 - W_B\left(1\right) - W_S\left(0\right)\right] > \kappa_B,$$

because $1 - W_B\left(1\right) > \underline{v}$ (since $v - W_B\left(v\right)$ is increasing in v from Lemma 1) and $p_B\left(\underline{v}\right) \geq W_S\left(0\right)$ (from Lemma 2). Similarly (9) implies

$$\ell_S\left(\zeta\right)\alpha_S\left[1 - W_B\left(1\right) - W_S\left(0\right)\right] > \kappa_S.$$

It follows that

$$1 - W_B\left(1\right) - W_S\left(0\right) > \max\left\{\frac{\kappa_B}{\ell_B\left(\zeta\right)\alpha_B}, \frac{\kappa_S}{\ell_S\left(\zeta\right)\alpha_S}\right\} \geq K\left(\zeta_0\right).$$

The last inequality is due to (7) in the proof of Lemma 3. Therefore $W_B\left(1\right) + W_S\left(0\right) < 1 - K\left(\zeta_0\right)$. ∎

Remark 2 *The bound in Proposition 1 is an upper bound for the ex-ante per capita utility of the arriving flow of buyers and sellers since*

$$\frac{1}{b+s}[b\int_0^1 W_B\left(v\right)dF\left(v\right) + s\int_0^1 W_S\left(c\right)dG\left(c\right)] \leq W_B\left(1\right) + W_S\left(0\right)$$
$$\leq 1 - K\left(\zeta_0\right).$$

From (7), $K\left(\zeta_0\right) = \min_{\zeta>0}\max\{\frac{\kappa_B}{\ell_B(\zeta)\alpha_B}, \frac{\kappa_S}{\ell_S(\zeta)\alpha_S}\}$, and therefore bargaining asymmetries negatively affect the welfare by increasing $K\left(\zeta_0\right)$ and moving the upper bound $1 - K\left(\zeta_0\right)$ closer to 0.

We now describe the main elements of the proof of the sufficiency part of Theorem 1; additional details are provided in the Appendix. As usual, we want to construct a mapping T such that its fixed point characterizes an (non-trivial steady-state) equilibrium, and prove that T has a fixed point. The mapping T is informally described as follows. Start with a pair of value functions (W_B, W_S) and a pair of mass functions (N_B, N_S), we construct best-response proposing strategies (p_B, p_S) and entry strategies (χ_B, χ_S). Then from those strategies and the original functions (W_B, W_S, N_B, N_S), we define a new pair of value functions (W_B^*, W_S^*) through the Bellman equations, and a new pair of mass functions (N_B^*, N_S^*) through the steady-state equations. Thus a fixed point of T (i.e. $(W_B, W_S, N_B, N_S) = (W_B^*, W_S^*, N_B^*, N_S^*)$) characterizes an equilibrium.

We will utilize the *Schauder fixed point theorem*: if D is a nonempty compact convex subset of a Banach space and T is a continuous function from D to D, then T has a fixed point. In order to make this theorem applicable, certain difficulties need to be overcome. The main difficulty is that as we apply the mapping T, we need to preserve positive entry. To deal with this difficulty, we first prove existence of what we call an *ε-equilibrium*, which is an actual equilibrium of the *ε-model* described below.

The ε-model differs from our original model in three ways. First, we ensure that all buyers with type $v \geq 1 - \varepsilon$ and all sellers with type $c \leq \varepsilon$ enter, even if they would make a loss. Such losses will be subsidized (or reimbursed) by a third party. Still, we may not have a positive lower bound for the mass of traders in the market because the outflow rate (i.e. $\ell_B q_B(v)$ or $\ell_S q_S(c)$) could be potentially very large. We impose the second modification, which ensures that the arrival rates ℓ_B and ℓ_S are bounded from above by some $\bar{\ell}$. We replace the matching function $M(B, S)$ with $\min\{M(B, S), B\bar{\ell}, S\bar{\ell}\}$, which clearly satisfies Assumption 2. This matching function implies $\ell_B \leq \bar{\ell}$ and $\ell_S \leq \bar{\ell}$.

While the first two modifications are made to make the mass of traders bounded from below, we also want it to be bounded from above, because our domain D needs to be compact. It suffices to have a lower bound for the outflow rate ($\ell_B q_B(v)$ or $\ell_S q_S(c)$). For a type who enters without subsidy, there is an upper bound for its mass because her expected trading surplus must be larger than her search cost. More precisely, for a participating v-buyer who is not subsidized, $\ell_B q_B(v) \geq \kappa_B$. However, a subsidized buyer could have $\ell_B q_B(v) < \kappa_B$. Our third modification is to disqualify subsidized traders in a way that ensures the outflow rates of subsidized types are at least κ_B or κ_S. The disqualification process is a Poisson process, with the rate equal to the minimum that makes the outflow rate at least κ_B or κ_S. Thus for any v-buyer, either subsidized or not, the gross outflow rate must be $\max\{\ell_B q_B(v), \kappa_B\}$.

In the Appendix, we formally define a mapping $T_\varepsilon : D_\varepsilon \rightarrow D_\varepsilon$, $(W_B, W_S, N_B, N_S) \longmapsto (W_B^*, W_S^*, N_B^*, N_S^*)$ (on an appropriate domain D_ε) such that its fixed point characterizes an ε-equilibrium; and show that T_ε has a fixed point E, i.e. $T_\varepsilon(E) = E$. That is, there exists an ε-equilibrium. In the next proposition, we show that if $\varepsilon > 0$ is small and $\bar{\ell}$ large, any ε-equilibrium is in fact an equilibrium of our original model.

Proposition 2 *Suppose $K(\zeta_0) < 1$. Then if $\varepsilon > 0$ is small enough and $\bar{\ell}$ large enough, any fixed point of T_ε characterizes a non-trivial steady-state equilibrium.*

The full proof of Proposition 2 is in the Appendix. Here we outline the basic argument. In an ε-equilibrium, there are two sets of marginal types. Let $\underline{v}$ be the lowest buyers' type who would enter without subsidy and let $\underline{v}^*$ be the lowest buyers' type who enters ($\underline{v}^* \leq 1 - \varepsilon$ by construction). Define $\bar{c}$ and $\bar{c}^*$ similarly. To claim that an ε-equilibrium is a true equilibrium, it suffices to show that $\underline{v}^* < 1 - \varepsilon$, $\bar{c}^* > \varepsilon$ and $\ell_B < \bar{\ell}$, $\ell_S < \bar{\ell}$.

First we claim that the entry gap $\underline{v}^* - \bar{c}^*$ cannot exceed $K(\zeta_0)$. A marginal buyer of type $\underline{v}^*$ cannot have the expected profit greater than his search cost. Moreover, such a marginal buyer can have a positive profit only when he proposes, because no seller would propose less than $\underline{v}^*$, for the same reason as in the original model. Since the price offer $\bar{c}^*$ is surely accepted, the expected profit conditional on matching is at least $\underline{v}^* - \bar{c}^*$. Therefore $\kappa_B \geq \ell_B(\zeta)\alpha_B(\underline{v}^* - \bar{c}^*)$. Applying the same logic to the sellers, we have $\kappa_S \geq \ell_S(\zeta)\alpha_S(\underline{v}^* - \bar{c}^*)$. Therefore

$$\underline{v}^* - \bar{c}^* \leq \min\left\{\frac{\kappa_B}{\ell_B(\zeta)\alpha_B}, \frac{\kappa_S}{\ell_S(\zeta)\alpha_B}\right\} \leq K(\zeta_0). \tag{10}$$

The last inequality is due to (7).

Second, we claim that in ε-equilibrium, the inflows of traders are approximately balanced, i.e. $b[1 - F(\underline{v})] \approx sG(\bar{c})$, when ε is small. From buyers' steady-state equation,

$$b[1 - F(\underline{v}^*)] = \int_{\underline{v}^*}^{1} \max\{\ell_B(\zeta)q_B(v), \kappa_B\}\, dN_B(v).$$

(recall that in ε-equilibrium, the outflow rate is $\max\{\ell_B(\zeta)q_B(v), \kappa_B\}$). If no buyer is subsidized, the outflow (i.e. the right-hand side) is simply the trading outflow $\ell_B(\zeta)\int_{\underline{v}}^{1} q_B(v)dN_B(v)$. Now consider the case in which some buyers are subsidized (which implies $\underline{v}^* = 1 - \varepsilon$). Using the inequality $dN_B(v) \leq [b\bar{f}/(\kappa_B)]dv$,

$$0 \leq b[1 - F(\underline{v}^*)] - \ell_B(\zeta)\int_{\underline{v}^*}^{1} q_B(v)dN_B(v) \leq b\bar{f}\varepsilon.$$

The same logic applied to the sellers' side implies

$$0 \leq sG(\bar{c}^*) - \ell_S(\zeta)\int_{0}^{\bar{c}^*} q_S(c)dN_S(c) \leq s\bar{g}\varepsilon.$$

Because the trading outflows must be balanced, i.e.

$$\ell_B(\zeta)\int_{\underline{v}^*}^{1} q_B(v)dN_B(v) = \ell_S(\zeta)\int_{0}^{\bar{c}^*} q_S(c)dN_S(c),$$

we have

$$|b[1 - F(\underline{v}^*)] - sG(\bar{c}^*)| \leq \max\{b\bar{f}, s\bar{g}\} \cdot \varepsilon. \tag{11}$$

If we let $\varepsilon \to 0$, then we have $b[1 - F(\underline{v}^*)] - sG(\bar{c}^*) \to 0$ from (11), while $\underline{v}^* - \bar{c}^*$ is bounded away from 1 according to $K(\zeta_0) < 1$ and (10). In the limit, we must have the strict inequalities $\bar{c}^* > 0$ and $\underline{v}^* < 1$. It follows that for all small enough $\varepsilon > 0$, we have $\bar{c}^* > \varepsilon$ and $\underline{v}^* < 1 - \varepsilon$. In such an ε-equilibrium with small ε, no trader is subsidized. Hence the marginal entrants must be able to recover their search costs. In particular, ζ is bounded away from 0 and ∞. Thus as long as $\bar{\ell}$ are chosen to be large enough, our modification of the matching function does not have a bite. It follows that we obtain a true equilibrium of our original model.

The existence part of Theorem 1 now follows from Proposition 1 and Proposition 2. The necessary and sufficient condition for existence of a non-trivial equilibrium turns out to be the same in the MW model, which differs from ours only in one respect: MW assume full information bargaining, i.e. bargainers know each other's type. Consequently, proposers hold their partners to their reservation values. We note that our general existence proof (Theorem 1) adapts with minor changes. The proof is even easier because we do not have to consider proposing strategies in our construction of the mapping T.

4   Full trade equilibria

A *full trade equilibrium* is defined as a non-trivial equilibrium in which every meeting results in trade: $q_B(v) = q_S(c) = 1$ for all $v \in [\underline{v}, 1]$ and $c \in [0, \bar{c}]$. The following properties of

a full trade equilibrium are immediate. First, the supports for active buyers' types and active sellers' types are separate, i.e. $\underline{v} > \bar{c}$. This is because otherwise a buyer with $\underline{v}$ will not trade if he meets a seller with $\bar{c}$: the seller will not propose or accept anything less than $\bar{c} + W_S(\bar{c}) = \bar{c}$, while the buyer will only propose or accept something strictly below $\underline{v} - W_B(\underline{v}) = \underline{v}$. Second, the lowest buyer's (and hence all active buyers') offer $p_B(\underline{v})$ is exactly at the level acceptable to all active sellers, i.e. $p_B(\underline{v}) = \bar{c}$; and similarly, the highest seller's (and hence all active sellers') offer $p_S(\bar{c})$ is exactly at the level acceptable to all active buyers: $p_S(\bar{c}) = \underline{v}$. It is easy to see that the converse is also true, i.e. if a non-trivial equilibrium has the above properties, then it is full-trade.

In a full trade equilibrium, the marginal type equations in Lemma 2 take the following form:

$$\ell_B(\zeta)\alpha_B(\underline{v} - \bar{c}) = \kappa_B, \tag{12}$$

$$\ell_S(\zeta)\alpha_S(\underline{v} - \bar{c}) = \kappa_S. \tag{13}$$

Noticing that $\ell_S(\zeta)/\ell_B(\zeta) = \zeta$, (12) and (13) can be easily solved for ζ and $\underline{v} - \bar{c}$:

$$\zeta = \zeta_0, \tag{14}$$

$$\underline{v} - \bar{c} = K(\zeta_0). \tag{15}$$

In steady state, the inflow of active buyers must equal the inflow of active sellers:

$$b[1 - F(\underline{v})] = sG(\bar{c}). \tag{16}$$

Since $\underline{v} - \bar{c}$ is determined from (15), $\underline{v}$ and $\bar{c}$ are uniquely pinned down by (16). It is easy to see that there is at most one full trade equilibrium.

Remark 3 *A full trade equilibrium, if exists, is uniquely characterized by equations (12), (13), and (16).*

Suppose the existence condition $K(\zeta_0) < 1$ holds. Then the above system has a solution with $\underline{v} < 1$ and $\bar{c} > 1$. We use subscript "0" to denote the objects of this unique full trade equilibrium candidate, e.g. $(\zeta_0, \underline{v}_0, \bar{c}_0)$.[7]

Although the condition $K(\zeta_0) < 1$ guarantees that a full trade equilibrium candidate exists, this candidate may not really be an equilibrium, since buyers may have an incentive to bid lower than $\bar{c}_0$, and similarly sellers may have an incentive to bid above $\underline{v}_0$.

We claim in the following proposition that if the discount rate is small, the equilibrium is unique and involves full trade. Together with the result in the previous section, it also implies existence of full trade equilibrium.

Proposition 3 *Suppose $K(\zeta_0) < 1$. A $\underline{r} > 0$ exists such that for all $r \in (0, \underline{r}]$, there is a unique equilibrium, and it involves full trade.*

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[7]Other endogenous variables are easily obtained. In particular, for $v \in [\underline{v}_0, 1]$ and $c \in [0, \bar{c}_0]$, $W_{B0}(v) = \frac{\ell_B(\zeta_0)}{r + \ell_B(\zeta_0)}(v - \underline{v}_0)$, $W_{S0}(c) = \frac{\ell_S(\zeta_0)}{r + \ell_S(\zeta_0)}(\bar{c}_0 - c)$, $N_{B0}(v) = \frac{b[F(v) - F(\underline{v}_0)]}{\ell_B(\zeta_0)}$, $N_{S0}(c) = \frac{sG(c)}{\ell_S(\zeta_0)}$.

Proof. In the Appendix. ■

The intuition for the uniqueness result is as follows. First, the slopes of reservation prices $v - W_B(v)$ and $c + W_S(c)$ become small when r is small (Lemma 1). Thus when r is small we cannot have $\underline{v} < \bar{c}$, otherwise we have e.g. $p_B(1) < 1 - W_B(1) < \bar{c}$, which is impossible to have in equilibrium because the price offer $p_B(1)$ is unacceptable to even marginal sellers. Second, when $\underline{v} > \bar{c}$, the demand and supply become very elastic as $r \to 0$ and this leads to marginal buyers unwilling to propose below $\bar{c}$ and marginal sellers unwilling to propose above $\underline{v}$, i.e. leads to a full trade equilibrium.

Appendix

Proof of Lemma 1.   We prove the results for buyers only. We use an argument from mechanism design. For any $v \in [0,1]$, define

$$t_B(v) \equiv \alpha_B \int_{\{c:p_B(v)-c\geq W_S(c)\}} p_B(v) \frac{dN_S(c)}{S} + \alpha_S \int_{\{c:v-p_S(c)\geq W_B(v)\}} p_S(c)\frac{dN_S(c)}{S}.$$

The buyers' Bellman equation (1) implies for any $v, \hat{v} \in [0,1]$ and any $\chi \in \{0,1\}$,

$$rW_B(v) \geq \chi \cdot \{\ell_B[q_B(\hat{v})v - t_B(\hat{v}) - q_B(\hat{v})W_B(v)] - \kappa_B\}$$

or equivalently $W_B(v) \geq \chi \cdot u_B(v,\hat{v})$ where

$$u_B(v,\hat{v}) \equiv \frac{\ell_B[q_B(\hat{v})v - t_B(\hat{v})] - \kappa_B}{r + \ell_B q_B(\hat{v})}.$$

And the inequality becomes equality if $\hat{v} = v$ and $\chi = \chi_B(v)$. Let $U_B(v) \equiv \max_{\hat{v}\in[0,1]} u_B(v,\hat{v})$. We then have $W_B(v) = \chi_B(v)u_B(v,v) = \chi_B(v)U_B(v) = \max\{U_B(v),0\}$. For any $\hat{v}$, $u_B(v,\hat{v})$ is affine and nondecreasing in v. Milgrom and Segal (2002) Envelope Theorem implies $U_B(v)$ is absolutely continuous, convex, nondecreasing, and with slope $\ell_B q_B(v)/(r + \ell_B q_B(v))$ whenever differentiable. The same properties are inherited by $W_B(v)$, except that its slope becomes $\chi_B(v)\ell_B q_B(v)/(r + \ell_B q_B(v))$.

Obviously $U_B(0) < 0$. Let $\underline{v} \equiv \sup\{v \in [0,1] : U_B(v) < 0\}$. By continuity of U_B, we have $\underline{v} > 0$ and $U_B(\underline{v}) \leq 0$. But $U_B(\underline{v}) < 0$ is impossible in nontrivial equilibrium because it implies $\chi_B(v) = 0$ $\forall v \in [0,1]$ and hence $B = 0$. Thus $U_B(\underline{v}) = W_B(\underline{v}) = 0$. By monotonicity of U_B, for all $v < \underline{v}$, we have $U_B(v) < 0$ and hence $\chi_B(v) = W_B(v) = 0$. Moreover, $q_B(v) > 0$ for all $v \geq \underline{v}$. It is because for all $v \geq \underline{v}$, the fact $U_B(v) \geq 0$ implies $\ell_B q_B(v) \geq \kappa_B > 0$. It furthermore implies $U'_B(\underline{v}+) \geq \ell_B q_B(\underline{v}+)/(r + \ell_B q_B(\underline{v}+)) > 0$. Thus for all $v > \underline{v}$, we have $U_B(v) > 0$ and hence $\chi_B(v) = 1$ and $W_B(v) = U_B(v)$. From steady-state equation (5), $[\underline{v},1]$ is the support of N_B. Since the inflow distributions F do not have atom points, neither does N_B. Hence $B > 0$ implies $\underline{v} < 1$. Finally, the convexity of U_B implies that q_B is nondecreasing on $[\underline{v},1]$. ■

Proof of Lemma 2.   *Step 1:* Suppose, by way of contradiction, $p_B(v) > \bar{c}$ for some $v \in [\underline{v},1]$. Then $p_B(v)$ is accepted by any active seller (because $c + W_S(c)$ is increasing in c by Lemma 1). A type v buyer can lower his offer without losing acceptance probability.

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But then $p_B(v)$ does not solve the proposing problem in (2). Therefore $p_B(v) \leq \bar{c}$ for all $v \in [\underline{v}, 1]$. Similarly $p_S(c) \geq \underline{v}$ for all $c \in [0, \bar{c}]$.

Step 2: A buyer with type $\underline{v}$ cannot get positive bargaining surplus when he is a responder, i.e. the second term inside the square bracket of (1), evaluated at $v = \underline{v}$, is 0. This is because, from step 1, $\underline{v} - W_B(\underline{v}) = \underline{v}$ is no higher than $p_S(c)$ proposed by any active seller. Then, since $W_B(\underline{v}) = 0$ from Lemma 1, the Bellman equation (1) evaluated at $v = \underline{v}$ implies $\ell_B(\zeta)\alpha_B \pi_B(\underline{v}) - \kappa_B = 0$. It follows that $\pi_B(\underline{v}) > 0$ and hence $\pi_B(v) > 0$ for all $v \in [\underline{v}, 1]$ (because any buyer can choose $p = p_B(\underline{v})$ in his proposing problem in (2)). Similarly, we can prove $\ell_S(\zeta)\alpha_S \pi_S(\bar{c}) - \kappa_S = 0$ and $\pi_S(c) > 0$ for all $c \in [0, \bar{c}]$.

Step 3: Fix any $v \in [\underline{v}, 1]$. From the inequality $\pi_B(v) > 0$ proved in step 2, we have $v - W_B(v) > p_B(v)$ and $p_B(v) \geq c + W_S(c)$ for some c. The last result is equivalent to $p_B(v) \geq W_S(0)$ because $c + W_S(c)$ is increasing in c. Similarly we can prove for all $c \in [0, \bar{c}]$, $c + W_S(c) < p_S(c) \leq 1 - W_B(1)$.

Step 4: Let $\Gamma(p)$ be the probability that price p will be accepted by a seller: $\Gamma(p) \equiv \int_{\{c: p-c \geq W_S(c)\}} dN_S(c)/S$. Obviously Γ is nondecreasing. Then the buyers' proposing problem in (2) can be written as $\pi_B(v) = \max_{p \in [0,1]}[v - W_B(v) - p]\Gamma(p)$. Pick any $v_1, v_2 \in [\underline{v}, 1]$, $v_2 > v_1$. Let $p_1 \equiv p_B(v_1)$ and $p_2 \equiv p_B(v_2)$. Revealed preference implies

$$[v_1 - W_B(v_1) - p_1]\Gamma(p_1) \geq [v_1 - W_B(v_1) - p_2]\Gamma(p_2) \tag{17}$$

and

$$[v_2 - W_B(v_2) - p_2]\Gamma(p_2) \geq [v_2 - W_B(v_2) - p_1]\Gamma(p_1).$$

Sum these two inequalities and then simplify. We obtain

$$[(v_2 - W_B(v_2)) - (v_1 - W_B(v_1))] \cdot [\Gamma(p_2) - \Gamma(p_1)] \geq 0.$$

Suppose, by the way of contradiction, $p_2 < p_1$. Then the above inequality implies $\Gamma(p_2) \geq \Gamma(p_1)$ and the monotonicity of Γ implies $\Gamma(p_2) \leq \Gamma(p_1)$. We thus have $\Gamma(p_2) = \Gamma(p_1) > 0$, where the last inequality is from step 2. Substitute back into (17), we have $p_2 \geq p_1$, a contradiction. ∎

We need some definitions and lemmas to prove the sufficiency part of Theorem 1.

Definition 2 *Let $\bar{\ell} > \max\{\kappa_B, \kappa_S\}$ and $\varepsilon \in (0, \bar{\varepsilon}]$, where*

$$\bar{\varepsilon} \equiv \min\{1, \frac{\bar{f}\bar{\ell}}{\kappa_B \underline{f}}, \frac{\bar{g}\bar{\ell}}{\kappa_S \underline{g}}\}.$$

Let $C[0,1]$ be the Banach space of real continuous bounded functions defined on $[0,1]$, endowed with the supremum norm. Define $D_\varepsilon \subset (C[0,1])^4$ as the set of all tuples (W_B, W_S, N_B, N_S) such that (i) W_B, N_B and N_S are nondecreasing, while W_S is nonincreasing, (ii) W_B, W_S, N_B and N_S have Lipschitz constants no greater than $\bar{\ell}/(r + \bar{\ell})$, $\bar{\ell}/(r + \bar{\ell})$, $b\bar{f}/\kappa_B$ and $s\bar{g}/\kappa_S$ respectively, and (iii) $W_B(0) = W_S(1) = N_B(0) = N_S(0) = 0$ and $N_B(1) \geq \varepsilon b\underline{f}/\bar{\ell}$, $N_S(1) \geq \varepsilon s\underline{g}/\bar{\ell}$.

Lemma 4 *The set D_ε is nonempty, convex and compact for any $\bar{\ell} > \max\{\kappa_B, \kappa_S\}$ and any $\varepsilon \in (0, \bar{\varepsilon}]$.*

Proof. Obviously, D_ε is convex and closed. To see D_ε is nonempty, let $W_B(v) = W_S(c) = 0$ for all v, c, and $N_B(v) = b\bar{f}v/\kappa_B$, $N_S = s\bar{g}c/\kappa_S$. Since $\varepsilon \le \bar{\varepsilon}$, we have $N_B(1) \ge \varepsilon b\underline{f}/\bar{\ell}$ and $N_S(1) \ge \varepsilon s\underline{g}/\bar{\ell}$. All other restrictions of D_ε are obviously satisfied, thus D_ε is nonempty. The set D_ε is a uniformly bounded family of functions on a compact set $[0, 1]$, and is also an equicontinuous family of functions because the Lipschitz constant for every function in D_ε is at most $\max\left\{1, b\bar{f}/\kappa_B, s\bar{g}/\kappa_S\right\}$. By Ascoli-Arzela Theorem (see e.g. Royden (1988) p.169), D_ε is compact. ∎

Definition 3 *The mapping $T_\varepsilon : D_\varepsilon \to D_\varepsilon$, $(W_B, W_S, N_B, N_S) \longmapsto (W_B^*, W_S^*, N_B^*, N_S^*)$ is defined through the following steps. The best-response proposals are defined as*

$$p_B(v) \equiv \max\left\{\arg\max_{p \in [0,1]} \int_{\{c:p-c \ge W_S(c)\}} [v - p - W_B(v)]dN_S(c)/S\right\} \qquad (18)$$

$$p_S(c) \equiv \min\left\{\arg\max_{p \in [0,1]} \int_{\{v:v-p \ge W_B(v)\}} [p - c - W_S(c)]dN_B(v)/B\right\}. \qquad (19)$$

The search value functions W_B^* and W_S^* are defined through rearranged Bellman equations (1) and (3):

$$W_B^*(v) \equiv \max_{\chi \in \{0,1\}} \chi \cdot \left\{\frac{\ell_B}{r + \ell_B}(\alpha_B \pi_B(v) + \alpha_S \int_{\{c:v-p_S(c) \ge W_B(v)\}} [v - p_S(c) - W_B(v)]\frac{dN_S(c)}{S}) \right.$$

$$\left. -\frac{\kappa_B}{r + \ell_B}\right\} + \frac{\ell_B}{r + \ell_B}W_B(v), \qquad (20)$$

$$W_S^*(c) \equiv \max_{\chi \in \{0,1\}} \chi \cdot \left\{\frac{\ell_S}{r + \ell_S}(\alpha_S \pi_S(c) + \alpha_B \int_{\{v:p_B(v)-c \ge W_S(c)\}} [p_B(v) - c - W_S(c)]\frac{dN_B(v)}{B}) \right.$$

$$\left. -\frac{\kappa_S}{r + \ell_S}\right\} + \frac{\ell_S}{r + \ell_S}W_S(c), \qquad (21)$$

where the arrival rates ℓ_B and ℓ_S are defined using the modified matching function: $\ell_B = \min\left\{M(B, S), B\bar{\ell}, S\bar{\ell}\right\}/B$, $\ell_S = \min\left\{M(B, S), B\bar{\ell}, S\bar{\ell}\right\}/S$ (and $B = N_B(1)$, $S = N_S(1)$). The entry strategies $\chi_B(v)$ and $\chi_S(c)$ are defined as the maximizers in (20) and (21) respectively; wherever multiple maximizers exist, we pick $\chi = 1$. Finally, the distributions of active trader types N_B^* and N_S^* are defined as

$$N_B^*(v) \equiv \int_0^v \frac{\chi_B^*(x)b}{\max\{\ell_B q_B(x), \kappa_B\}}dF(x) \qquad (22)$$

where $\chi_B^*(v)$ is 1 if $\chi_B(v) = 1$ or $v \ge 1 - \varepsilon$, and is 0 otherwise; and

$$N_S^*(c) \equiv \int_0^c \frac{\chi_S^*(x)s}{\max\{\ell_S q_S(x), \kappa_S\}}dG(x) \qquad (23)$$

where $\chi_S^*(c)$ is 1 if $\chi_S(c) = 1$ or $c \le \varepsilon$, and is 0 otherwise.

We now show that the definition of T_ε is legitimate, i.e. it is well-defined and $T_\varepsilon(D_\varepsilon) \subset D_\varepsilon$. Pick any $E \equiv (W_B, W_S, N_B, N_S) \in D_\varepsilon$. By construction $N_B(1)$, $N_S(1) > 0$, so that ℓ_B and ℓ_S are well-defined. Second, $N_B(v)$ and $\tilde{v}(v) \equiv v - W_B(v)$ are continuous in v; $N_S(c)$ and $\tilde{c}(c) \equiv c + W_S(c)$ are continuous in c. Third, $\tilde{v}$ and $\tilde{c}$ are strictly increasing (since $r > 0$). It follows that the objective functions in (18) and (19) are continuous in p. Therefore the $\arg\max$ correspondence in (18) and (19) are nonempty-valued and compact-valued. Thus p_B and p_S are well-defined. Now it is obvious that all other constructed objects, in particular $W_B^*, W_S^*, N_B^*, N_S^*$, are well-defined.

It remains to verify that $(W_B^*, W_S^*, N_B^*, N_S^*) \in D_\varepsilon$. First, by our construction $W_B^*, W_S^*, N_B^*, N_S^*$ are absolutely continuous; and whenever differentiable,

$$W_B^{*\prime}(v) = \chi_B(v) \frac{\ell_B}{r + \ell_B} q_B(v) \left[1 - W_B'(v)\right] + \frac{\ell_B}{r + \ell_B} W_B'(v),$$

$$W_S^{*\prime}(c) = -\chi_S(c) \frac{\ell_S}{r + \ell_S} q_S(c) \left[1 + W_S'(c)\right] + \frac{\ell_S}{r + \ell_S} W_S'(c),$$

$$N_B^{*\prime}(v) \equiv \frac{\chi_B^*(v) \, b f(v)}{\max\{\ell_B q_B(v), \kappa_B\}}, \quad N_S^{*\prime}(c) \equiv \frac{\chi_S^*(c) \, s g(c)}{\max\{\ell_S q_S(c), \kappa_S\}}.$$

From these derivatives we see $(W_B^*, W_S^*, N_B^*, N_S^*)$ satisfies the conditions (i) and (ii) in Definition 3. Second, it is easy to verify that $(W_B^*, W_S^*, N_B^*, N_S^*)$ also satisfies the condition (iii) in Definition 3. Therefore $(W_B^*, W_S^*, N_B^*, N_S^*) \in D_\varepsilon$. We conclude that Definition 3 of T_ε is legitimate.

The following lemma will be used to prove the continuity of T_ε.

Lemma 5 *Let $\{\Phi_n\}$ be a sequence of continuous c.d.f.'s with supports contained in $[0, 1]$ and $\{\psi_n\}$ a sequence of real functions on $[0, 1]$. Suppose $\{\Phi_n\}$ is uniformly convergent to some c.d.f. Φ; $\{\psi_n\}$ is convergent to some real function ψ almost everywhere on $[0, 1]$; and the absolute values and total variations of $\{\psi_n\}$ and ψ are bounded by some constant C. Then*

$$\lim_{n \to \infty} \int_0^1 \psi_n(x) \, d\Phi_n(x) = \int_0^1 \psi(x) \, d\Phi(x).$$

Proof. For each n, since ψ_n is of bounded variation and Φ_n is continuous, hence ψ_n is Riemann integrable with respect to Φ_n (see e.g. Apostol (1974) p.159 Theorem 7.27 and p.144 Theorem 7.6). Similarly, ψ is of bounded variation and Φ (as the uniform limit of a sequence of continuous functions) is continuous, hence ψ is Riemann integrable with respect to Φ. Moreover,

$$\left| \int_0^1 \psi_n d\Phi_n - \int_0^1 \psi d\Phi \right| \leq \left| \int_0^1 \psi_n d\Phi_n - \int_0^1 \psi_n d\Phi \right| + \left| \int_0^1 \psi_n d\Phi - \int_0^1 \psi d\Phi \right|.$$

The first part of the right-hand side can be written, through integration by parts for Riemann-Stieltjes integrals (see e.g. Apostol (1974) p.144 Theorem 7.6), as $\left| \int [\Phi - \Phi_n] \, d\psi_n \right|$ and hence is bounded by $C \cdot \sup_{x \in [0,1]} |\Phi(x) - \Phi_n(x)|$, which converges to 0 as $n \to \infty$, due to the uniform convergence of $\{\Phi_n\}$. The second part also converges to 0 as $n \to \infty$, due to Lebesgue's dominated convergence theorem (see e.g. Apostol (1974) p.270 Theorem 10.27). $\blacksquare$

Lemma 6 *The mapping* $T_\varepsilon : D_\varepsilon \to D_\varepsilon$ *is continuous for any* $\bar\ell > \max\{\kappa_B, \kappa_S\}$ *and any* $\varepsilon \in (0, \bar\varepsilon]$.

Proof. Fix $(r, \bar\ell) \gg (0, \max\{\kappa_B, \kappa_S\})$ and $\varepsilon \in (0, \bar\varepsilon]$. We write the constructed objects in Definition 3 as functions of $E \equiv (W_B, W_S, N_B, N_S)$ explicitly, e.g. $B(E)$, $\ell_B(E)$, $p_B(v, E)$, $W_B(v, E)$, $N_B(v, E)$ etc. We need to show that: for any sequence $\{E_n\}$ on D_ε, $E_n \to E$ implies $T_\varepsilon(E_n) \to T_\varepsilon(E)$. (Recall that we use the uniform metric on D_ε.)

Step 1. Obviously $B(E)$, $S(E)$, $\ell_B(E)$ and $\ell_S(E)$ are continuous in E.

Step 2. It is easy to see that: $I[p \geq c + W_S(c)]$ (where $I[\cdot]$ is 1 if the condition inside the bracket holds, and 0 otherwise), as a function of (c, p, E), is continuous on $\{(c, p, E) : p \neq c + W_S(c)\}$. Similarly, $I[p \leq v - W_B(v)]$, as a function of (v, p, E), is continuous on $\{(v, p, E) : p \neq v - W_B(v)\}$.

Step 3. $\hat\pi_B(v, p, E) \equiv [v - p - W_B(v)] \int_0^1 I[p \geq c + W_S(c)] \, dN_S(c)/S(E)$ is continuous in (v, p, E). To see this, let $(v_n, p_n, E_n) \to (v, p, E)$. Then firstly $v_n - p_n - W_{Bn}(v_n) \to v - p - W_B(v)$ (note that the convergence $W_{Bn} \to W_B$ is uniform); from step 2, $I[p_n \geq c + W_{Sn}(c)] \to I[p \geq c + W_S(c)]$ except at the c such that $p = c + W_S(c)$ (note that there is at most one such c since $r > 0$ and $E \in D_\varepsilon$ imply $c + W_S(c)$ is strictly increasing). Applying Lemma 5, we obtain $\hat\pi_B(v_n, p_n, E_n) \to \hat\pi_B(v, p, E)$. Thus $\hat\pi_B(v, p, E)$ is continuous. Similarly, $\hat\pi_S(c, p, E) \equiv [p - c - W_S(c)] \int_0^1 I[p \leq v - W_B(v)] \, dN_B(v)/B(E)$ is continuous in (c, p, E).

Step 4. From step 3 and Berge's maximum theorem, $\pi_B(v, E)$ (which is equal to $\max_{p \in [0,1]} \hat\pi_B(v, p, E)$) is continuous in (v, E), and $P_B(v, E) \equiv \arg\max_{p \in [0,1]} \hat\pi_B(v, p, E)$ is nonempty-valued, compact-valued, and upper-hemicontinuous in (v, E).

Analogous results can be proved for $\pi_S(c, E)$ and $P_S(c, E) \equiv \arg\max_{p \in [0,1]} \hat\pi_S(c, p, E)$.

Step 5. $p_B(v, E)$ is continuous on $\{(v, E) : P_B(v, E) \text{ is a singleton}\}$. To see this, let $(v_n, E_n) \to (v, E)$ and let $p_B(v_n, E_n) \to p$. Then from step 4, $p \in P_B(v, E)$. Thus, if $p \neq p_B(v, E)$ then $P_B(v, E)$ is not a singleton. Moreover, $p_B(v, E)$ is continuous on $\{(v, E) : v - W_B(v) > W_S(0)\}$. An analogous result can be proved for p_S.

Step 6. Let $E \in D_\varepsilon$ and $E_n \to E$. Then $p_B(v, E_n) \to p_B(v, E)$ a.e. $v \in [0, 1]$. To see this, first consider v satisfying $v - W_B(v) < W_S(0)$. Then it is easy to see that $\pi_B(v, E_n) = 0 = \pi_B(v, E)$ and $P_B(v, E_n) = [0, W_{Sn}(0)] = P_B(v, E)$. Thus $p_B(v, E_n) \to W_S(0) = p_B(v, E)$. Now consider v satisfying $v - W_B(v) > W_S(0)$. By a standard revealed preference argument, any selection of $P_B(\cdot, E)|_{\{v : v - W_B(v) > W_S(0)\}}$ is nondecreasing. It follows that, for all but countably many v's in $\{v : v - W_B(v) > W_S(0)\}$, $P_B(v, E)$ is a singleton. Then $p_B(v, E_n) \to p_B(v, E)$ a.e. from step 5. An analogous result can be proved for p_S.

Step 7. Let $E \in D_\varepsilon$ and $E_n \to E$. Then, from steps 1, 2, 4, 6, and Lemma 5, $W_B^*(v, E_n) \to W_B^*(v, E)$ $\forall v$ and $W_S^*(c, E_n) \to W_S^*(c, E)$ $\forall c$.

Step 8. It is easy to see that $\chi_B(v, E)$ is continuous on $\{(v, E) : \ell_B(E) \Pi_B(v, E) \neq \kappa_B\}$, where $\Pi_B(v, E)$ is the expression inside the square brackets in (20). Furthermore, given E, there is at most one v such that $\ell_B(E) \Pi_B(v, E) = \kappa_B$. To see this, notice that $\ell_B(E) \Pi_B(v, E)$ is nondecreasing in v, and if $\ell_B(E) \Pi_B(v, E) = \kappa_B$ then $\ell_B(E) q_B(v, E) \geq \kappa_B$ and hence $\frac{\partial}{\partial v}[\ell_B(E) \Pi_B(v, E)] = \ell_B(E) q_B(v, E)[1 - W_B'(v)] \geq \frac{\kappa_B r}{r + \ell_B(E)} > 0$.

As a result, given any $E \in D_\varepsilon$, if $E_n \to E$ then $\chi_B(v, E_n) \to \chi_B(v, E)$ a.e. $v \in [0, 1]$. Obviously χ_B^* has the same property, and analogous results can be proved for χ_S and χ_S^*.

Step 9. Let $E \in D_\varepsilon$ and $E_n \to E$. Then, from steps 1, 2, 6, and Lemma 5, $q_B(v, E_n) \to q_B(v, E)$ a.e. $v \in [0, 1]$, and $q_S(c, E_n) \to q_S(c, E)$ a.e. $c \in [0, 1]$. This together with step 8 implies that $N_B^*(v, E_n) \to N_B^*(v, E)$ $\forall v$ and $N_S^*(c, E_n) \to N_S^*(c, E)$ $\forall c$, again due to Lemma 5.

Step 10. Let $E \in D_\varepsilon$ and $E_n \to E$. From steps 7 and 9, $W_B^*(\cdot, E_n)$, $W_S^*(\cdot, E_n)$, $N_B^*(\cdot, E_n)$ and $N_S^*(\cdot, E_n)$ converge pointwise to $W_B^*(\cdot, E)$, $W_S^*(\cdot, E)$, $N_B^*(\cdot, E)$ and $N_S^*(\cdot, E)$ respectively. Moreover, the pointwise convergence is equivalent to uniform convergence, because each of those function sequences form an equicontinuous family of functions on a compact domain $[0, 1]$ (see e.g. Royden (1988) p.168). We therefore conclude that $T_\varepsilon(E_n) \to T_\varepsilon(E)$. ∎

Lemma 7 *Fix any $\bar{\ell} > \max\{\kappa_B, \kappa_S\}$ and any $\varepsilon \in (0, \bar{\varepsilon}]$. There exists some $E \in D_\varepsilon$ such that $T_\varepsilon(E) = E$.*

Proof. As claimed before, D_ε is a nonempty, convex and compact set in a Banach space $(C[0, 1])^4$ and the mapping T_ε is continuous. Then we obtain our result by applying the Schauder Fixed Point Theorem (which is stated in the text). ∎

Proof of Proposition 2. Suppose $E = (W_B, W_S, N_B, N_S) \in D_\varepsilon$ is a fixed point of T_ε. Then E, together with the constructed objects through the transformation from E to $T_\varepsilon(E)$, constitutes what we call an ε-equilibrium. Moreover, an ε-equilibrium satisfies all the equilibrium conditions in Definition 1 if one can verify that:

(i) The traders that might have been forced into the market in fact enter voluntarily: $\underline{v}^* \equiv \inf\{v : \chi_B^*(v) = 1\} < 1 - \varepsilon$, and $\bar{c}^* \equiv \sup\{c : \chi_S^*(c) = 1\} > \varepsilon$;

(ii) No trader disqualification is necessary: $\ell_B q_B(v) \geq \kappa_B$ if $\chi_B^*(v) = 1$, and $\ell_S q_S(c) \geq \kappa_S$ if $\chi_S^*(c) = 1$;

(iii) Market tightness ζ does not become too small or too large, so that $\ell_B, \ell_S < \bar{\ell}$.

The following steps 1-6 will show that, for any $\bar{\ell} > \max\{\kappa_B, \kappa_S\}$, any $\varepsilon \in (0, \bar{\varepsilon}]$, and any associated fixed point of T_ε, several equilibrium properties hold. Then steps 7 and 8 will show that (i)-(iii) also hold if $\varepsilon > 0$ is small enough and $\bar{\ell}$ large enough.

Step 1. $E \in D_\varepsilon$ implies $v - W_B(v)$ and $c + W_S(c)$ are strictly increasing. Thus, from (18) and (21), we have $p_B(v) \leq \bar{c}^* + W_S(\bar{c}^*)$ and $p_S(c) \geq \underline{v}^* - W_B(\underline{v}^*)$.

Step 2. The expression inside the curly bracket in (20) can be written as

$$\frac{\ell_B}{r + \ell_B}\left[\alpha_B \pi_B(v) + \alpha_S \int \max\{v - p_S(c) - W_B(v), 0\}\frac{dN_S(c)}{S} - \frac{\kappa_B}{\ell_B}\right], \qquad (24)$$

which is continuous in v. Then by the definition of $\underline{v}^*$, $\chi = 0$ is a maximizer in (18) when $v = \underline{v}^*$. In other words, (24) is non-positive when $v = \underline{v}^*$. Now evaluate (20) at $v = \underline{v}^*$. From the above result and that $W_B^* = W_B$, we have $W_B(\underline{v}^*) = \frac{\ell_B}{r + \ell_B}W_B(\underline{v}^*)$, or $W_B(\underline{v}^*) = 0$.

Step 3. The fact that (24) is non-positive when $v = \underline{v}^*$ implies $\alpha_B \pi_B(\underline{v}^*) \leq \kappa_B/\ell_B$. Applying similar logic to the sellers, we also have $W_S(\bar{c}^*) = 0$ and $\alpha_S \pi_S(\bar{c}^*) \leq \kappa_S/\ell_S$.

Step 4. Notice that $\pi_B(\underline{v}^*) \geq \underline{v}^* - \bar{c}^*$ since the choice variable p in the definition (2) of π_B can be taken as $\bar{c}^*$. Similarly $\pi_S(\bar{c}^*) \geq \underline{v}^* - \bar{c}^*$. Then step 3 implies

$$\underline{v}^* - \bar{c}^* \leq \min\left\{\frac{\kappa_B}{\ell_B \alpha_B}, \frac{\kappa_S}{\ell_S \alpha_S}\right\} \leq K(\zeta_0). \tag{25}$$

Step 5. The expression inside the curly brackets in (20), which can be written as (24), is increasing in v. A single-crossing argument implies that χ_B and χ_B^* are increasing. Therefore, if $v \geq \underline{v}$, then (24) is non-negative, which implies $\ell_B q_B(v) \geq \kappa_B$. Similarly, χ_S and χ_S^* are decreasing, and for any $c \leq \bar{c} \equiv \sup\{c : \chi_S(c) = 1\}$, we have $\ell_S q_S(c) \geq \kappa_S$.

Step 6. Equation (22), $N_B^* = N_B$, and step 5 imply

$$b[1 - F(\underline{v}^*)] - \int_{\underline{v}^*}^1 \ell_B q_B(v) \, dN_B(v) = \int_{\underline{v}^*}^1 \max\{0, \kappa_B - \ell_B q_B(v)\} \, dN_B(v). \tag{26}$$

The r.h.s. of (26) is clearly non-negative. Moreover, it is also no greater than $b\bar{f}\varepsilon$. To see this, consider two (exhaustive) cases: $\underline{v}^* = \underline{v}$ and $\underline{v}^* < \underline{v}$. First consider the case $\underline{v}^* = \underline{v}$. From step 5 the r.h.s. of (26) is 0. Next, consider the case $\underline{v}^* < \underline{v}$. From the definitions of $\underline{v}^*$ and $\underline{v}$, we have $\underline{v}^* = 1 - \varepsilon$. The r.h.s. of (26) is no greater than $b\bar{f}\varepsilon$ because $N_B'(v) \leq \frac{b\bar{f}}{\kappa_B}$. Similar logic can be applied to the sellers' side. Therefore we obtain

$$0 \leq b[1 - F(\underline{v}^*)] - \int_{\underline{v}^*}^1 \ell_B q_B(v) \, dN_B(v) \leq b\bar{f}\varepsilon$$

$$0 \leq sG(\bar{c}^*) - \int_0^{\bar{c}^*} \ell_S q_S(c) \, dN_S(c) \leq s\bar{g}\varepsilon.$$

On the other hand, by definition of ℓ_B, q_B, ℓ_S, q_S, we have

$$\int_{\underline{v}^*}^1 \ell_B q_B(v) dN_B(v) = \int_0^{\bar{c}^*} \ell_S q_S(c) dN_S(c).$$

Therefore,

$$|b[1 - F(\underline{v}^*)] - sG(\bar{c}^*)| \leq \max\{b\bar{f}, s\bar{g}\} \cdot \varepsilon. \tag{27}$$

Step 7. The previous six steps work with a particular fixed point of T_ε given $(\varepsilon, \bar{\ell})$. In this and the next step, we let $(\varepsilon, \bar{\ell}) \to (0, \infty)$ and consider an associated sequence of fixed points. Along any subsequence, $\bar{c}^*$ cannot approach 0 because otherwise (27) implies $\underline{v}^* \to 1$ and hence $\underline{v}^* - \bar{c}^* \to 1$, violating (25) and $K(\zeta_0) < 1$. Similarly, $\underline{v}^*$ cannot approach 1 along any subsequence. Therefore, in the tail of the sequence, we have $\bar{c}^* > \varepsilon$ and $\underline{v}^* < 1 - \varepsilon$, i.e. (i) holds. Notice that (i) implies $\underline{v}^* = \underline{v}$ and $\bar{c}^* = \bar{c}$. Thus step 5 implies (ii) also holds in the tail.

Step 8. From steps 5 and 7, we have $\ell_B(\zeta) \geq \kappa_B$ and $\ell_S(\zeta) \geq \kappa_S$ in the tail as $(\varepsilon, \bar{\ell}) \to (0, \infty)$. Thus $\zeta \equiv B/S$ is bounded away from 0 and ∞. It follows that, in the tail, $\ell_B, \ell_S < \bar{\ell}$, i.e. (iii) holds. ∎

The sufficiency part of Theorem 1 follows from Lemma 7 and Proposition 2.

The proof of our uniqueness result (Proposition 3) is expedited by the following lemma that bounds from below the entry gap $\underline{v} - \bar{c}$.

Lemma 8 *In any non-trivial steady-state equilibrium, we have*

$$\underline{v} - \bar{c} \geq \frac{\kappa - r}{r + \kappa} \left[1 - W_B(1) - W_S(0) \right]. \tag{28}$$

where $\kappa \equiv \min\{\kappa_B, \kappa_S\}$.

Proof. Since q_B is nondecreasing (from Lemma 1),

$$q_B(v) \geq q_B(\underline{v}) \geq \alpha_B \int_{\{c:p_B(\underline{v})-c \geq W_S(c)\}} \frac{dN_S(c)}{S}$$

for any $v \geq \underline{v}$. Next, we must have $\underline{v} > p_B(\underline{v}) \geq W_S(0)$ from Lemma 2. The first inequality is from Lemma 2. The second inequality is from the proof of Lemma 1. Then from the buyers' marginal type equation (in Lemma 2), we obtain $\ell_B q_B(v) \left[\underline{v} - W_S(0) \right] \geq \kappa_B \ \forall v \geq \underline{v}$. From Lemma 1, we have

$$\frac{d}{dv} \left[v - W_B(v) \right] = \frac{r}{r + \ell_B q_B(v)} \leq \frac{r}{r + \kappa_B/(\underline{v} - W_S(0))}.$$

Hence

$$1 - W_B(1) - \underline{v} = \int_{\underline{v}}^1 \frac{d}{dv} \left[v - W_B(v) \right] dv \leq \frac{r}{r + \kappa_B/(\underline{v} - W_S(0))},$$

$$\frac{1 - W_B(1) - \underline{v}}{\underline{v} - W_S(0)} \leq \frac{r}{(\underline{v} - W_S(0))r + \kappa_B} < \frac{r}{\kappa_B}.$$

The last inequality is due to $\underline{v} > p_B(\underline{v})$ and $r > 0$. Rearranging, we get

$$\frac{1 - W_B(1) - \underline{v}}{1 - W_B(1) - W_S(0)} < \frac{r}{r + \kappa_B} \leq \frac{r}{r + \kappa}. \tag{29}$$

Repeating the previous arguments with the roles of buyers and sellers interchanged, we get

$$\frac{\bar{c} - W_S(0)}{1 - W_B(1) - W_S(0)} < \frac{r}{r + \kappa}. \tag{30}$$

Summing up (29) and (30) and rearranging terms yield the desired inequality. ∎

Proof of Proposition 3. We have already noted (in Remark 3) that there cannot be more than one full trade equilibrium. It suffices to prove that, if r is small, then in any (non-trivial steady-state) equilibrium, $p_B(v) = \bar{c} \ \forall v \geq \underline{v}$ and $p_S(c) = \underline{v} \ \forall c \leq \bar{c}$. We will only consider $r < \kappa \equiv \min\{\kappa_B, \kappa_S\}$, which through Lemma 1 implies $\underline{v} > \bar{c}$ in equilibrium. Now pick any equilibrium and focus on sellers. Since p_S is nondecreasing on $[0, \bar{c}]$ and $p_S(c) \geq \underline{v} \ \forall c \leq \bar{c}$ (from Lemma 2), the condition $p_S(c) = \underline{v} \ \forall c \leq \bar{c}$ is reduced to $p_S(\bar{c}) = \underline{v}$, or equivalently $p = \underline{v}$ is the only maximizer of $\max_{p \in [0,1]} \hat{\pi}_S(\bar{c}, p)$, where

$$\hat{\pi}_S(\bar{c}, p) = (p - \bar{c}) \int_{\underline{v}}^1 I\left[p \leq v - W_B(v)\right] \frac{dN_B(v)}{B}.$$

Since $\hat{\pi}_S(\bar{c}, p)$ is absolutely continuous in p, it is differentiable in p almost everywhere. Notice that $\partial \hat{\pi}_S(\bar{c}, p)/\partial p$ is 1 if $p < \underline{v}$, so that any $p < \underline{v}$ is never optimal. Proposing

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$p > 1 - W_B(1)$, which implies $\hat{\pi}_S(\bar{c}, p) = 0$, is also never optimal. If $\underline{v} < p < 1 - W_B(1)$, whenever differentiable, we have

$$\frac{\partial \pi_S(\bar{c}, p)}{\partial p} = 1 - \tilde{\Phi}(p) - (p - \bar{c}) \tilde{\phi}(p), \tag{31}$$

where $\tilde{\Phi}(p) \equiv \int_{\underline{v}}^{1} I[p \leq v - W_B(v)] \, dN_B(v)/B$ and $\tilde{\phi}(p) \equiv \tilde{\Phi}'(p)$. Define $\tilde{v}(v) \equiv v - W_B(v)$ and $\phi(x) \equiv N_B'(x)/B$. The function $\tilde{v}$ is strictly increasing since $r > 0$ (from Lemma 1), so that its inverse function $\tilde{v}^{-1}$ is well-defined on the range of $\tilde{v}$, and is also strictly increasing. Then

$$\tilde{\phi}(p) = \frac{\phi(\tilde{v}^{-1}(p))}{\tilde{v}'(\tilde{v}^{-1}(p))} \quad \forall p \in [\underline{v}, 1 - W_B(1)].$$

We want to show that, when $\underline{v} < p < 1 - W_B(1)$ the r.h.s. of (31) must be negative for all sufficiently small $r > 0$. From $r < \kappa$, Lemma 8 and 1, we obtain

$$p - \bar{c} > \underline{v} - \bar{c} \geq K(\zeta_0) \left(\frac{\kappa - r}{r + \kappa} \right) > 0. \tag{32}$$

Moreover, for all $v \geq \underline{v}$, we have $\tilde{v}'(v) = \frac{r}{r + \ell_B q_B(v)}$ (from Lemma 1) and $\ell_B q_B(v) \geq \kappa_B$ (from Lemma 2). Thus $\tilde{v}'(v) \leq r/(r + \kappa)$, and hence

$$\tilde{\phi}(p) \geq \left(1 + \frac{\kappa}{r} \right) \phi(\tilde{v}^{-1}(p)). \tag{33}$$

We now derive a lower bound for the market probability density of buyers' types ϕ. From the steady-state condition (5),

$$\phi(v) = \frac{bf(v)}{M(B, S) q_B(v)} \geq \frac{b\underline{f}}{M(B, S)} \quad \forall v \geq \underline{v} \tag{34}$$

and

$$B = \int_{\underline{v}}^{1} \frac{bf(v) \, dv}{\ell_B q_B(v)} < \frac{b}{\kappa_B} \equiv \bar{B}.$$

Similarly (6) implies

$$S < \frac{s}{\kappa_S} \equiv \bar{S}.$$

Since $M(B, S)$ is nondecreasing in each of its arguments, $M(B, S) \leq M(\bar{B}, \bar{S})$. Substituting this bound into (34) we obtain

$$\phi(v) \geq \frac{b\underline{f}}{M(\bar{B}, \bar{S})} \equiv \underline{\phi} \quad \forall v \geq \underline{v}. \tag{35}$$

Applying (32), (33) and (35) to (31), and simplifying, we find that for almost all $p \in [\underline{v}, 1 - W_B(1)]$,

$$\frac{\partial \pi_S(\bar{c}, p)}{\partial p} < 1 - K(\zeta_0) \left(\frac{\kappa}{r} - 1 \right) \underline{\phi}.$$

Similarly, $p_B(\underline{v}) \in [W_S(0), \bar{c}]$, and for almost all $p \in [W_S(0), \bar{c}]$, we have

$$\frac{\partial \pi_B(\underline{v}, p)}{\partial p} > -1 + K(\zeta_0)\left(\frac{\kappa}{r} - 1\right)\underline{\gamma}$$

where

$$\pi_B(\underline{v}, p) \equiv (\underline{v} - p) \int_0^{\bar{c}} I[p \geq c + W_S(c)]\frac{dN_S(c)}{S},$$

$$\underline{\gamma} \equiv \frac{s\underline{g}}{M\left(\bar{\bar{B}}, \bar{S}\right)}.$$

Therefore, if $1 - K(\zeta_0)\left(\frac{\kappa}{r} - 1\right)\underline{\phi} \leq 0$ and $-1 + K(\zeta_0)\left(\frac{\kappa}{r} - 1\right)\underline{\gamma} \geq 0$, or equivalently $r \leq \underline{r}$ where

$$\underline{r} \equiv \kappa \cdot \frac{K(\zeta_0)\min\left\{\underline{\phi}, \underline{\gamma}\right\}}{1 + K(\zeta_0)\min\left\{\underline{\phi}, \underline{\gamma}\right\}}, \tag{36}$$

we have $r < \kappa$, $\partial \pi_S(\bar{c}, p)/\partial p < 0$ for almost every $p \in (\underline{v}, 1 - W_B(1))$ and $\partial \pi_B(\underline{v}, p)/\partial p > 0$ for almost every $p \in (W_S(0), \bar{c})$. Hence $p_S(\bar{c}) = \underline{v}$ and $p_B(\underline{v}) = \bar{c}$. ∎

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