Implementation of Nash Bargaining Solutions with Non-Convexity*

Cheng-Zhong Qin† Guofu Tan‡ Adam Chi Leung Wong§

February 5, 2019

Abstract

Nash solutions for two-player bargaining problems with non-convexity are shown to be dictatorial selections of Nash product maximizers in recent literature. In this paper we show that these solutions are implementable as unique subgame perfect equilibrium payoff allocations of a sequential game.

Keywords: Bargaining problem, Non-convexity, Nash solution, Implementation

JEL Classification: C71, C72, C78

1 Introduction

We consider a class of two-player economic bargaining environments that naturally result in non-convex bargaining problems. By results in Naumova and Yanovskaya (2001) and Peters and Vermeulen (2012), solutions under the Nash axioms on this class are each dictatorial selections of Nash product maximizers. We show that each

*We gratefully acknowledge helpful comments from Youngsub Chun, Pradeep Dubey, Mamoru Kaneko, Abraham Neyman, Hans Peters, Yair Tauman, Shmuel Zamir, Yongsheng Xu, Junjie Zhou, and seminar and conference participants at the Hong Kong University of Science and Technology, the University of California, Santa Barbara, 2015 International Conference on Industrial Organization at Zhejiang University, June 17-18, Hangzhou China, 2015 Asian Game Theory Conference, August 24-26, Tokyo, and the 28th International Conference On Game Theory, Stony Brook, NY, July 17-21, 2017. Declarations of interest: none.

†Department of Economics, University of California, Santa Barbara, CA 93106. Email: qin@econ.ucsb.edu.

‡Department of Economics, University of Southern California, Los Angeles, CA 90089-0253. Email: guofutan@usc.edu.

§Corresponding author. Department of Economics, Lingnan University, 8 Castle Peak Road, Tuen Mun, Hong Kong. Email: adamwong@ln.edu.hk.

of these Nash solutions is implemented as the unique subgame perfect equilibrium payoff allocation of a sequential game based on our bargaining environment.

The sequential game has a *nomination phase* and a *trial phase*. The basic building block of the trial phase is the two-period version of Ståhl's (1972) alternating offer game. In the trial phase, the players sequentially bid in terms of delay time for being the first to make an offer.[1] In the nomination phase, the players simultaneously nominate candidate alternatives. Each player can only nominate one feasible alternative, and only the status quo and the nominated candidate choices are the alternatives on the table in the subsequent trial phase. We show that the resulting *nomination-trial game* implements the Nash solution in favor of the player who submits bid first in the trial phase, with the relative bargaining powers reflecting the players' relative patience.

2 Preliminaries

2.1 Bargaining environment

Two players, 1 and 2, are endowed with a compact set X (in a topological space) of alternatives that they can jointly achieve with agreement and a status quo $q \in X$, which the players will end up getting in case of disagreement. No randomization device is feasible for the players. However, delays are possible and time is continuous. A *feasible bargaining outcome* is a pair $\langle x, t \rangle$ with $x \in X$ and $t \in [0, \infty]$, which is understood as an agreement for choosing alternative x being reached after a delay of time t. The preferences of player $i \in \{1, 2\}$ over outcomes $\langle x, t \rangle$ are represented by the utility function

$$e^{-r_i t} U_i(x) + (1 - e^{-r_i t}) U_i(q),$$

where $r_i > 0$ and $U_i : X \to \mathbb{R}$ are respectively the instantaneous discount rate and utility function of player i. Set $\mathbf{r} = (r_1, r_2)$ and $U(\cdot) = (U_1(\cdot), U_2(\cdot))$. We assume that U is continuous and $U(x) \gg U(q)$ for some $x \in X$. We refer to each quadruple $(X, q, \mathbf{r}, U)$ satisfying the above assumptions as a *bargaining environment*.

2.2 Nash bargaining solutions

Following the axiomatic approach initiated by Nash (1950), a (two-player) bargaining problem is composed of a set $S \subseteq \mathbb{R}^2$ of feasible payoff allocations and a disagreement

[1] If the two players submit bid simultaneously, this trial phase can be regarded as a variant of the game in Moulin (1984) that implements the Kalai-Smorodinsky (1975) solution with convexity. The latter game cannot implement a single-valued solution in the presence of non-convexity, unless sequential bidding is considered.

point $d \in S$. A bargaining problem (S, d) is *regular* if S is compact and there exists $u \in S$ such that $u \gg d$; it is **r**-*star-shaped* (relative to the disagreement point) if[2]

$$u \in S \text{ and } t \in [0, \infty] \Rightarrow \left(e^{-r_i t} u_i + \left(1 - e^{-r_i t}\right) d_i\right)_{i \in \{1,2\}} \in S.$$

A *bargaining solution* on a domain $\mathcal{B}$ of bargaining problems is a mapping $f : \mathcal{B} \to \mathbb{R}^2$ such that $f(S, d) = (f_1(S, d), f_2(S, d)) \in S$ for all $(S, d) \in \mathcal{B}$.

Given a bargaining environment $(X, q, \mathbf{r}, U)$, the set of feasible payoff allocations is

$$S = \left\{ \left(e^{-r_i t} U_i(x) + \left(1 - e^{-r_i t}\right) d_i\right)_{i \in \{1,2\}} : x \in X \text{ and } t \in [0, \infty] \right\} \tag{1}$$

and the disagreement point is

$$d = U(q). \tag{2}$$

Note that a bargaining problem (S, d) generated by any bargaining environment with discount rates $\mathbf{r}$ is regular and $\mathbf{r}$-star-shaped. Conversely, any regular and $\mathbf{r}$-star-shaped problem can be derived from some bargaining environment with discount rates $\mathbf{r}$. Let $\mathcal{B}_{\mathbf{r}}$ denote the family of all regular and $\mathbf{r}$-star-shaped bargaining problems. Note that the problems in $\mathcal{B}_{\mathbf{r}}$ need not be convex so that there can be multiple Nash product maximizers.

Definition 1. A bargaining solution f on a domain $\mathcal{B}$ is said to be a *(single-valued) Nash (bargaining) solution* if there exist $i \in \{1, 2\}$ and $\alpha \in \mathbb{R}^2_{++}$ with $\alpha_1 + \alpha_2 = 1$ such that for any $(S, d) \in \mathcal{B}$,

$$\{f(S, d)\} = \operatorname*{argmax}_{u \in \Sigma(S,d)} u_i$$

where $\Sigma(S, d)$ is the set of Nash product maximizers of (S, d) with respect to bargaining powers α, i.e.,

$$\Sigma(S, d) \equiv \operatorname*{argmax}_{u \in S, u \geq d} (u_1 - d_1)^{\alpha_1} (u_2 - d_2)^{\alpha_2}.$$

The axiomatic foundation of Definition 1 is provided by Naumova and Yanovskaya (2001) and Peters and Vermeulen (2012). Their results show that a multi-valued n-player bargaining solution satisfying certain "Nash axioms" must iteratively maximize Nash products with distinct distributions of bargaining powers. Qin et al. (2017) identify the exact representations of single-valued Nash bargaining solutions. Namely, for a rich variety of domains including $\mathcal{B}_{\mathbf{r}}$ for any $\mathbf{r} \in \mathbb{R}_{++}$, the Nash solutions given by Definition 1 are the only single-valued bargaining solutions satisfying the

[2]In the special case with $r_1 = r_2$, (S, d) being $\mathbf{r}$-star-shaped means that S is star-shaped about d, i.e., for all $u \in S$ and $p \in [0, 1]$, $pu + (1 - p)d \in S$.

axioms of Invariance to Positive Affine Transformations,[3] Independence of Irrelevant Alternatives,[4] and Strict Individual Rationality.[5]

3 Implementation

Let $(X, q, \mathbf{r}, U)$ be a bargaining environment as introduced in Section 2. We consider non-cooperative implementation of Nash solutions to the associated bargaining problem. To this end, let $\Gamma(i, t, Y)$ denote the two-period alternating offer game as in Ståhl (1972), where $i \in \{1, 2\}$ is the first mover, $t \in [0, \infty]$ is the length of time delay between two offers, and $Y \subseteq X$ with $q \in Y$ is the set of alternatives on the table. Negotiation takes place according to the following *nomination-trial game*.

Phase 1: The Nomination Phase

- Each player i nominates a candidate alternative $x_i \in X$ simultaneously.

Phase 2: The Trial Phase

- Player 1 names a delay time $t_1 \in [0, \infty]$.
- After observing t_1, player 2 names another delay time $t_2 \in [0, \infty]$ with $t_2 \neq t_1$.
- If $t_1 < t_2$, the two players play $\Gamma(1, \frac{t_1+t_2}{2}, \{x_1, x_2, q\})$; if $t_2 < t_1$, they play $\Gamma(2, \frac{t_1+t_2}{2}, \{x_1, x_2, q\})$.

We adopt pure strategy subgame perfect equilibrium (hereafter SPE) as the equilibrium concept.

Our main result shows that the above nomination-trial game SPE-implements the Nash solution in favor of the player who submits bid first in the trial phase, with the relative bargaining powers determined by the players' relative patience.

Theorem 1. *Given any bargaining environment $(X, q, \mathbf{r}, U)$, the nomination-trial game has at least one SPE; the payoff allocation in any SPE is $f(S, d)$ where S, d are given by (1) and (2) and f is the Nash solution given in Definition 1 with domain $\mathcal{B}_{\mathbf{r}}$, $i = 1$, $\alpha_1 = r_2/(r_1 + r_2)$, and $\alpha_2 = r_1/(r_1 + r_2)$.*

Osborne and Rubinstein (1990) describe approximate implementations by various strategic models, including Rubinstein's (1982) alternating offer game with vanishing risk of breakdown and time discounting (Binmore et al., 1986) and the perturbed Nash demand game due to Nash (1953). These strategic models were primarily

[3] It means, for any $(S, d) \in \mathcal{B}$ and for any positive affine transformation $\tau : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\tau(S), \tau(d)) = \tau(f(S, d))$.

[4] It means, for any $(S, d), (T, d) \in \mathcal{B}$ with $S \subseteq T$, $f(T, d) \in S$ implies $f(S, d) = f(T, d)$.

[5] It means, for any $(S, d) \in \mathcal{B}$, $f(S, d) \gg d$.

constructed for dealing with convex bargaining problems. They do not implement single-valued bargaining solutions when allowing for non-convexity.[6]

4 Proof of Theorem 1

Due to translation invariance, we normalize the disagreement point to $\mathbf{0}$ without loss of generality. With this normalization, each player i's utility function becomes $e^{-r_i t}U_i(x)$. We first focus on the case of symmetric discounting.

Lemma 1. *Assume $r_1 = r_2 \equiv r$ and let $\delta \equiv e^{-r} \in (0,1)$ be the common discount factor per unit time. Let $t \in [0,\infty]$ and $a, b \in X$. Then, $\Gamma(i, t, \{a, b, q\})$ has at least one SPE. If $U(a), U(b) \geq \mathbf{0}$, then (i) $\Gamma(i, t, \{a, b, q\})$ has $\langle a, 0\rangle$ as its unique SPE outcome when $U_j(a) > \delta^t U_j(b)$ and $U_i(a) > U_i(b)$; (ii) $\Gamma(i, t, \{a, b, q\})$ has $\langle b, 0\rangle$ as its unique SPE outcome when $U_j(a) < \delta^t U_j(b)$ and $U_i(b) > 0$; (iii) the set of SPE outcomes of $\Gamma(i, t, \{a, b, q\})$ is $\{\langle a, 0\rangle, \langle b, 0\rangle\}$ when $U_j(a) = \delta^t U_j(b)$ and $U_i(a) \geq U_i(b)$.*

Proof. Given $t \in [0,\infty]$ and $a, b \in X$, $\Gamma(i, t, \{a, b, q\})$ is a finite game of perfect information; hence, it has at least one (pure strategy) SPE as can be shown by backward induction.

With $U_j(a) > \delta^t U_j(b)$ and $U_i(a) > U_i(b)$, it is optimal for player j to accept a whenever it is offered in $\Gamma(i, t, \{a, b, q\})$. Thus, since $U_i(a) > U_i(b)$, it is optimal for player i to offer a in SPE. This establishes (i). With $U_j(a) < \delta^t U_j(b)$ and $U_i(b) > 0$, player j always rejects any offer other than b in $\Gamma(i, t, \{a, b, q\})$. Thus, since $U_i(b) > 0$, it is optimal for player i to offer b which will be accepted by player j in SPE. This establishes (ii).

Now suppose $U_j(a) = \delta^t U_j(b)$ and $U_i(a) \geq U_i(b)$. The following describes a SPE in $\Gamma(i, t, \{a, b, q\})$ leading to outcome $\langle a, 0\rangle$: player i offers a and player j accepts a and b but rejects q in the first period; player j offers b and player i accepts any offer in the second period regardless of what happened in the first period. Similarly, the following describes a SPE in $\Gamma(i, t, \{a, b, q\})$ leading to outcome $\langle b, 0\rangle$: player i offers b and player j accepts b but rejects a and q in the first period; player j offers b and player i accepts any offer in the second period whatever happened in the first period. It is straightforward to show that there can be no other (pure strategy) SPE outcome. ■

Lemma 2. *Theorem 1 holds when $r_1 = r_2$.*

[6]For stationary SPE of Rubinstein's (1982) game, see Herrero (1989). With non-stationary SPE allowed, Rubinstein's game can even lead to non-vanishing delay when the risk of breakdown and time discounting vanish. See Binmore (1987).

Proof. Let $r = r_1 = r_2$ and $\delta = e^{-r} \in (0,1)$. Consider the trial phase following nominations $a, b \in X$. Without loss of generality we assume $U(a), U(b) \geq \mathbf{0}$. Any nominated candidate that is not individually rational can be replaced by the status quo q. We complete the remaining proof in five steps.

Step 1. If $U(a) = U(b)$, then the set of SPE outcomes of the trial phase is $\{\langle a, 0\rangle, \langle b, 0\rangle\}$, but the SPE payoff allocation is unique and equal to $U(a)$.

Step 2. If $U_1(a)U_2(a) > U_1(b)U_2(b)$, the trial phase has $\langle a, 0\rangle$ as the unique SPE outcome.

To see this, first note that the result holds if in addition $U_1(a) > U_1(b)$ and $U_2(a) > U_2(b)$. Now suppose that $U_j(a) \leq U_j(b)$ and $U_i(a) > U_i(b)$. Then

$$0 \leq \frac{U_i(b)}{U_i(a)} < \frac{U_j(a)}{U_j(b)} \leq 1.$$

From player i's point of view, $\langle a, 0\rangle$ is the unique best possible outcome in the trial phase. Player i can guarantee outcome $\langle a, 0\rangle$ by specifying t_i such that

$$0 \leq \frac{U_i(b)}{U_i(a)} < \delta^{t_i} < \frac{U_j(a)}{U_j(b)} \leq 1.$$

If player j chooses $t_j > t_i$, then the two players will subsequently play $\Gamma(i, t, \{a, b, q\})$ where $t = (t_i + t_j)/2 > t_i$. In this case, $\delta^t U_j(b) < U_j(a)$. Thus, since $U_i(a) > U_i(b)$, Lemma 1(i) implies that the unique SPE outcome of $\Gamma(i, t_i, \{a, b, q\})$ is $\langle a, 0\rangle$. If player j chooses $t_j < t_i$, then two players will subsequently play $\Gamma(j, t, \{a, b, q\})$ where $t = (t_i + t_j)/2 < t_i$. In this case, $\delta^t > \delta^{t_i}$ which in turn implies $U_i(b) < \delta^t U_i(a)$. Since $U_1(a)U_2(a) > U_1(b)U_2(b)$, $U(a) \geq 0$, and $U(b) \geq 0$, it follows that $U_j(a) > 0$. Thus, by interchanging i with j and a with b, Lemma 1(ii) implies that the unique SPE outcome of $\Gamma(j, t, \{a, b, q\})$ is $\langle a, 0\rangle$. Therefore, any SPE outcome of the trial phase must be $\langle a, 0\rangle$. Also, it is straightforward to see that the trial phase has SPE.

Step 3. If $U_1(a)U_2(a) = U_1(b)U_2(b) > 0$ and $U_1(a) > U_1(b)$, the trial phase has $\langle a, 0\rangle$ as the unique SPE outcome.

To see this, first note that the assumptions imply

$$0 < \frac{U_1(b)}{U_1(a)} = \frac{U_2(a)}{U_2(b)} < 1.$$

From player 1's point of view, $\langle a, 0\rangle$ is the unique best possible outcome in the trial phase. Player 1 can guarantee outcome $\langle a, 0\rangle$ by specifying t_1 such that $\delta^{t_1} = U_2(a)/U_2(b)$. Then, depending on player 2's choice of $t_2 \in [0, \infty]$ the players play either $\Gamma(1, t, \{a, b, q\})$ with $t = (t_1 + t_2)/2 > t_1$ or $\Gamma(2, t, \{a, b, q\})$ with $t = (t_1 + t_2)/2 < t_1$. In the first case, it follows from Lemma 1(i) (with $i = 1$ and $j = 2$) that the unique SPE outcome of $\Gamma(1, t, \{a, b, q\})$ is $\langle a, 0\rangle$. In the second case, it follows from Lemma 1(ii) (with a, b interchanged, $i = 2$, and $j = 1$) that the unique SPE outcome of $\Gamma(2, t, \{a, b, q\})$ is again $\langle a, 0\rangle$. Therefore, any SPE outcome of the trial phase must be $\langle a, 0\rangle$. Also, it is straightforward to see that the trial phase has SPE.

For the remaining proof, we consider the overall game. Let $u = f(S, \mathbf{0}) \gg \mathbf{0}$ where f is the Nash solution given by Definition 1 with $i = 1$ and $\alpha = (1/2, 1/2)$. Pick $a \in X$ such that $U(a) = u$.

Step 4. u is a SPE payoff allocation.

To see this, consider a strategy profile, in which each player selects $a \in X$ in the nomination phase and following each pair of nominations, they play a SPE strategy profile in the trial phase. By Step 1, the outcome of this strategy profile is $\langle a, 0\rangle$ and the resulting payoff allocation is u. It remains to show that no one has incentives to deviate from a in the nomination phase. Since $U(a) = f(S, \mathbf{0})$, if a player unilaterally deviates from a to any $b \in X$ with $U(b) \geq \mathbf{0}$, the conditions in one of the previous three steps would be satisfied and then the SPE payoff allocation of trial phase would still be $U(a)$. Therefore, no player has incentives to deviate.

Step 5. u is the only SPE payoff allocation.

To see this, first note that player 1's payoff in any SPE is at least u_1 because player 1 can guarantee herself this payoff by nominating a. Similarly, player 2's payoff in any SPE is at least u_2. Since $u = f(S, \mathbf{0})$ is on the strict Pareto frontier of S, u must be the only SPE payoff allocation. ■

From the classical analysis of Fishburn and Rubinstein (1982), the assumption $r_1 = r_2$ is without loss of generality because any asymmetry in discounting can be incorporated in the instantaneous utility functions. More specifically, recall that player i's utility function is $e^{-r_i t} U_i(x)$; thus, for any $\bar{r} > 0$, player i's preferences can also be represented by the utility function

$$\left[e^{-r_i t} U_i(x)\right]^{\bar{r}/r_i} = e^{-\bar{r}t} \left(U_i(x)\right)^{\bar{r}/r_i} = e^{-\bar{r}t} \tilde{U}_i(x),$$

where

$$\tilde{U}_i(x) \equiv \left(U_i(x)\right)^{\bar{r}/r_i}.$$

Hence, $\bar{r}$ can be regarded as the common discount rate, if we let $\tilde{U}_i$ be player i's instantaneous utility function.

Proof of Theorem 1. Transform the utility functions as above. By Lemma 2, the nomination-trial game SPE-implements the Nash solution given in Definition 1 with $i = 1$ and $\alpha = (1/2, 1/2)$. The objective function in the first-round maximization in Definition 1 is

$$\left(\tilde{U}_1(x)\right)^{1/2} \left(\tilde{U}_2(x)\right)^{1/2} = \left(U_1(x)\right)^{\bar{r}/2r_1} \left(U_2(x)\right)^{\bar{r}/2r_2} = \left[\left(U_1(x)\right)^{\alpha_1} \left(U_2(x)\right)^{\alpha_2}\right]^{\bar{r}(r_1+r_2)/2r_1 r_2}.$$

Maximizing $\left(\tilde{U}_1(x)\right)^{1/2} \left(\tilde{U}_2(x)\right)^{1/2}$ is equivalent to maximizing $\left(U_1(x)\right)^{\alpha_1} \left(U_2(x)\right)^{\alpha_2}$. ■

References

Binmore, K., 1987. Perfect equilibria in bargaining models. In: Binmore, K., Dasgupta, P. (Eds.), The Economics of Bargaining. Basil Blackwell, Oxford, Ch. 5, pp. 77–105.

Binmore, K., Rubinstein, A., Wolinsky, A., 1986. The Nash bargaining solution in economic modelling. The RAND Journal of Economics 17 (2), 176–188.

Fishburn, P. C., Rubinstein, A., 1982. Time preference. International Economic Review 23 (3), 677–694.

Herrero, M. J., 1989. The Nash program: non-convex bargaining problems. Journal of Economic Theory 49 (2), 266–277.

Kalai, E., Smorodinsky, M., 1975. Other solutions to Nash's bargaining problem. Econometrica 43 (3), 513–518.

Moulin, H., 1984. Implementing the Kalai-Smorodinsky bargaining solution. Journal of Economic Theory 33 (1), 32–45.

Nash, J. F., 1950. The bargaining problem. Econometrica 18 (2), 155–162.

Nash, J. F., 1953. Two-person cooperative games. Econometrica 21 (1), 128–140.

Naumova, N., Yanovskaya, E., 2001. Nash social welfare orderings. Mathematical Social Sciences 42 (3), 203–231.

Osborne, M., Rubinstein, A., 1990. Bargaining and markets. Academic Press San Diego.

Peters, H., Vermeulen, D., 2012. WPO, COV and IIA bargaining solutions for non-convex bargaining problems. International Journal of Game Theory 41 (4), 851–884.

Qin, C.-Z., Tan, G., Wong, A. C. L., 2017. Characterization and implementation of Nash bargaining solutions with non-convexity, uSC-INET Research Paper No. 17-30. Available at SSRN: http://dx.doi.org/10.2139/ssrn.3052015.

Rubinstein, A., 1982. Perfect equilibrium in a bargaining model. Econometrica 50 (1), 97–109.

Ståhl, I., 1972. Bargaining theory. Stockholm: Economics Research Institute, Stockholm School of Economics.