The Rate of Convergence to Perfect Competition of Matching and Bargaining Mechanisms

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Abstract

We study the steady state of a market with incoming cohorts of buyers and sellers who are matched pairwise and bargain under private information. A friction parameter is τ, the length of the time period until the next meeting. We provide a necessary and sufficient condition for the convergence of mechanism outcomes to perfect competition at the linear rate in τ, which is shown to be the fastest possible among all bargaining mechanisms. The condition requires that buyers and sellers always retain some bargaining power. The bargaining mechanisms that satisfy this condition are called nonvanishing bargaining power (NBP) mechanisms. Simple random proposer take-it-or-leave-it protocols are NBP, while k-double auctions (k-DA) are not. We find that k-DAs have equilibria that converge to perfect competition at a linear rate, converge at a slower rate or even do not converge at all.

Keywords: Matching and Bargaining, Search, Double Auctions, Foundations for Perfect Competition, Rate of Convergence

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1 Introduction

A number of papers on dynamic matching and bargaining have shown that, as frictions vanish, equilibria converge to perfect competition.[1] But it is also important to know how

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[1] Papers that address convergence include Rubinstein and Wolinsky (1985), Gale (1986), Gale (1987), Rubinstein and Wolinsky (1990), Mortensen and Wright (2002), and, with private information, Butters (1979), Wolinsky (1988), De Fraja and Sakovics (2001), Serrano (2002), Moreno and Wooders (2002), Lauermann (2009), Satterthwaite and Shneyerov (2007), Atakan (2009).

rapidly the equilibria converge. To our knowledge, this question has not been addressed in the literature.

In contrast, the rate of convergence to efficiency has been the focus of the literature on static double auctions. It is important to know how large n needs to be so that we can call a double auction with n buyers and sellers approximately competitive. Rustichini, Satterthwaite, and Williams (1994) show robust convergence of double-auction equilibria in the symmetric class at the fast rate $O(1/n)$ for the bid/offer strategies and the super-fast rate $O(1/n^2)$ for the ex ante traders' welfare, where n is the number of traders in the market.[2] Moreover, the double auction converges at the rate that is fastest among all incentive-compatible and individually rational mechanisms (Satterthwaite and Williams (2002); Tatur (2005)). Cripps and Swinkels (2005) substantially enrich the model by allowing correlation among bidders' valuations, and show convergence at the rate $O(1/n^{2-\varepsilon})$, where $\varepsilon > 0$ is arbitrarily small.[3]

For a dynamic matching and bargaining market, the question of how small frictions need to be for equilibria to be approximately competitive is equally important. In this paper, we fill this gap by proving a rate of convergence result for a decentralized model of trade. We study the steady state of a market with incoming cohorts of buyers and sellers who are randomly matched pairwise and bargain without knowing each other's reservation value. The model is in discrete time and shares several features with the model in Satterthwaite and Shneyerov (2007). Exactly as in that paper, a friction parameter is τ, the length of the time period until the next meeting. There are per-period participation costs, $\tau\kappa_B$ for buyers and $\tau\kappa_S$ for sellers. There is also time discounting at the instantaneous rate r.[4]

Our model is different from Satterthwaite and Shneyerov (2007) in that we consider pairwise matching and general trading mechanisms. (Satterthwaite and Shneyerov (2007) restrict attention to auctions.) Atakan (2009) provides an important extension of the results of Satterthwaite and Shneyerov (2007) to multiple units and allows each trader to be a proposer with certain probability. Atakan (2009) allows the proposers to offer direct bargaining mechanisms (DMBs, as defined in Myerson and Satterthwaite (1983)), and shows that in equilibrium they can do no better than simply make price offers.[5] With this justification, he confines the analysis to take it or leave it price offer games.[6]

We consider a class of DBMs, *nonvanishing bargaining power (NBP) mechanisms*, that generalize certain properties of the random-proposer take it or leave it games described above. As the name suggests, the NBP conditions require that each trader has at least some bargaining power even when $\tau \to 0$. When a buyer with valuation v meets a seller with cost c, there is an expected bargaining surplus $U(v, c)$ available to them, over and

[2] See also Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), Satterthwaite (1989), Williams (1991).

[3] Reny and Perry (2006) allow interdependent values and show that it is almost efficient and almost fully aggregates information as $n \to \infty$, but do not address the rate of convergence issue.

[4] Satterthwaite and Shneyerov (2007) also discuss the rate of convergence to perfect competition, and the relevance of making τ small. However, there is no general rate of convergence result, but only within a class of full trade equilibria. They also discuss an interpretation of the inverse of τ as a measure of local market size. We do not repeat these discussions here.

[5] This parallels the no haggling result in Riley and Zeckhauser (1983).

[6] Shneyerov and Wong (forthcoming) derive necessary and sufficient conditions for the existence of non-trivial market equilibria for this protocol.

above their market values of search. For each active buyer of type v, we require that his interim utility $U_B(v)$ are always at least some (independent of τ) fraction $\alpha_B > 0$ of the *minimal* bargaining surplus $\min_c U(v,c)$ in his meetings in the market. The condition for the sellers is parallel. Together with incentive compatibility and ex post individual rationality of the outcomes, these conditions define NBP mechanisms.

Our paper's convergence results can be summarized as follows. We show in Theorem 1 (and Corollary 1) that the NBP conditions are *necessary and sufficient* for an incentive compatible and ex post individually rational DBM to have nontrivial market equilibria convergent to the Walrasian limit at a linear rate as $\tau \to 0$.[7] In addition, we also derive an explicit bound on the inefficiency of any nontrivial market equilibrium. We argue that the bound can be tight when frictions are small.

Using a notion of asymptotic optimality inspired by Satterthwaite and Williams (2002), we show that the NBP mechanisms are asymptotically optimal: in terms of the welfare, the rate of convergence cannot be faster for any individually rational DBM (Theorem 2). Our notion of asymptotic optimality differs from that in Satterthwaite and Williams (2002). There it means that the rate of convergence is the fastest possible for some distributions of traders' types (worst-case asymptotic optimal). Our notion means that the rate of convergence is the fastest possible for any distributions of traders' types.[8]

The equilibria of several interesting protocols lead to NBP mechanisms and therefore converge at the fastest possible rate. We show that this is the case for the random-proposer take it or leave it protocol described above. We also show that this is the case for several other protocols where each trader can guarantee that he or she can make a take it or leave it offer with a positive probability.

One popular protocol that has equilibrium outcomes that are not NBP is the k-double auction (k-DA). We find that k-DA equilibria can be either convergent at a linear rate, convergent at a slower rate or even divergent. Our double auction result can be compared to the findings in Serrano (2002). In a dynamic setting, Serrano (2002) studies a mechanism that in some respects resembles a double auction (the set of bids is restricted to be a finite grid) and finds that "as discounting is removed, equilibria with Walrasian and non-Walrasian features persist".[9] Serrano points out, however, that "after removing the finite sets of traders' types and of allowed prices, the present model confirms Gale's one-price result and has a strong Walrasian flavor". We, on the other hand, find that in our model, non-convergent equilibria exist even if the bargaining protocol is the "unrestricted" double auction.

Lauermann (2009) also adopts a general mechanism approach in a setting with exogenous exit of traders and *no* search cost as in Satterthwaite and Shneyerov (2008), and proposes a set of necessary and sufficient conditions on limit market outcomes that guarantee convergence to perfect competition. Lauermann (2009) does not address the issue of the rate of convergence. Our k-DA examples show that the rate of convergence can be arbitrarily slow. At the same time, we completely characterize the sequences of mechanisms

[7] A trivial, uninteresting equilibrium in which none of the traders enter always exists.

[8] In our model, each trader's type is distributed on $[0,1]$ interval and has a nonvanishing density there.

[9] This simplified bargaining mechanism was introduced in Wolinsky (1990) and also used in Blouin and Serrano (2001).

that converge at the fastest possible rate, which is shown to be linear in τ.[10]

The structure of the paper is as follows. Section 2 introduces the model. Section 3 derives basic properties of equilibria. Section 4 contains our main convergence results. Section 5 contains the asymptotic optimality results. Section 6 contains the k-double auction counterexamples. Section 7 concludes. The proofs are in the Appendix.

2 The Model

The agents are potential buyers and sellers of a homogeneous, indivisible good. Each buyer has a unit demand, while each seller has a unit supply. All traders are risk neutral. Potential buyers are heterogeneous in their valuations (or types) v of the good. Potential sellers are also heterogeneous in their costs (or types) c of providing the good. The buyers draw their types i.i.d. from some strictly increasing c.d.f. F and the sellers draw their types i.i.d. from some strictly increasing c.d.f. G. For simplicity, we assume that the supports of these distributions are $[0, 1]$. Each trader's type remains the same over time. We index time periods by $t = \cdots, -1, 0, 1, \cdots$; each time period has length $\tau > 0$. The instantaneous discount rate is $r \geq 0$, and the corresponding discount factor is $R_\tau = e^{-r\tau}$. Each period consists of the following stages.

- The mass b of potential buyers and the mass s of potential sellers are born. The new-born buyers draw their types i.i.d. from the distribution F and the new-born sellers draw their types i.i.d. from the distribution G.

- Entry (or participation, or being active): The new-born potential buyers and sellers decide whether to enter the market. Those who enter together with the current pools of traders in the market compose the set of active traders. Those who do not enter leave the market immediately, and get zero payoff.

- The active buyers and sellers incur participation costs $\tau\kappa_B$ and $\tau\kappa_S$ respectively.

- The active buyers and sellers are randomly matched in pairs. The mass of the matches is given by a matching function $M(B, S)$, where B and S are the masses of active buyers and active sellers currently in the market. The probability ℓ_B that a buyer is matched is equal to $M(B, S)/B$, and he is equally likely to meet any active seller. Symmetrically, the seller's matching probability is $\ell_S = M(B, S)/S$ and she is equally likely to meet any active buyer.[11] The matching is anonymous.

[10] As Shneyerov and Wong (forthcoming) have recently emphasized, dynamic matching and bargaining games with *costly* search as here have very different properties from the games with costless search.

[11] We conjecture that such a matching process can arise in the limit of finite economies. Let $\mathbb{G}_L^{n_B,n_S}$ be the set of all bilateral matching graphs with n_B buyer nodes, n_S seller nodes and L edges. All the edges are assumed to be nonadjacent. Let $\mathbb{P}_L^{n_B,n_S}$ be the probability measure on $\mathbb{G}_L^{n_B,n_S}$ that assigns equal probability to each graph in $\mathbb{G}_L^{n_B,n_S}$. In this stochastic matching model, (i) the probability of matching is L/n_B for the buyers and L/n_S for the seller, and (ii) conditional on matching, each buyer is equally likely to be matched with any seller. Our matching technology can be understood in this framework, as arising in the limit as $n \to \infty$ of the measures $\mathbb{P}_{M(n_B,n_S)}^{n_B,n_S}$ with n_B approximately equal to $n \cdot B$ and n_S approximately equal to $n \cdot S$. If the matching function exhibits constant returns to scale, it is easy to see that a buyer's probability of matching $\ell_B = M(n_B, n_S)/n_B \approx M(B, S)/B$, and similarly $\ell_S \approx M(B, S)/B$. Moreover, each buyer continues to be equally likely matched with any seller.

- The bargaining outcome of the match between type v buyer and type c seller is determined by an incentive compatible (IC) and ex post individually rational (IR) direct bargaining mechanism (DBM). If the buyer and seller trade, they leave the market. If the bargaining breaks down, both traders remain in the market.

Assumption 1 (matching function) *The matching function M is continuous on $\mathbb{R}^2_+$, nondecreasing in each argument, exhibits constant returns to scale (i.e. is homogeneous of degree one), and satisfies $M(0,S) = M(B,0) = 0$ and $M(B,S) \leq \min\{B,S\}$.*

Let $\zeta \equiv B/S$ be the ratio of the mass of buyers to the mass of sellers currently active in the market (or market tightness), and define $m(\zeta) \equiv M(\zeta, 1)$. Since the matching technology exhibits constant returns to scale, the matching probabilities for buyers and sellers are

$$\ell_B(\zeta) \equiv \frac{m(\zeta)}{\zeta}, \quad \ell_S(\zeta) \equiv m(\zeta).$$

Note that ℓ_B and ℓ_S are continuous on $\mathbb{R}_{++}$, and respectively nonincreasing and nondecreasing functions of ζ. Define

$$K(\zeta) \equiv \frac{\kappa_B}{\ell_B(\zeta)} + \frac{\kappa_S}{\ell_S(\zeta)}.$$

The function $K(\zeta)$ can be interpreted as the participation costs incurred by a pair of traders over the time period of length $\tau = 1$, inflated by their probabilities of matching $\ell_B(\zeta)$ and $\ell_S(\zeta)$.

Our equilibrium notion parallels that of Satterthwaite and Shneyerov (2007), so we skip many details elucidated there and focus on the differences due to the fact that we are considering DBMs. We assume that the market is in a steady state and denote the market distributions of *active* buyer and seller types as Φ and Γ. We denote the supports of these distributions as A_B and A_S. Denote as $W_B(v)$ and $W_S(c)$ the beginning-of-period market utilities of type v buyers and type c sellers, contingent on entry. Only the buyers with $W_B(v) \geq 0$ and sellers with $W_S(c) \geq 0$ are active, so A_B and A_S are the sets $\{v \in [0,1] : W_B(v) \geq 0\}$ and $\{c \in [0,1] : W_S(c) \geq 0\}$. Let $\underline{v} = \inf A_B$ be the lower boundary of A_B and $\bar{c} = \sup A_S$ be the upper boundary of A_S. We only consider equilibria with entry, i.e. those in which $\underline{v} < 1$ and $\bar{c} > 0$, so that the steady-state masses B and S of active buyers and sellers are positive.

As in Satterthwaite and Shneyerov (2007), the market utilities are taken as exogenous to the DBM, because they simply reflect the values of outside options. If we normalize the no trade outcome as yielding 0 utilities to the traders, the relevant reservation values become

$$\tilde{v}(v) = v - R_\tau W_B(v), \quad \tilde{c}(c) = c + R_\tau W_S(c). \tag{1}$$

Following Satterthwaite and Shneyerov (2007), we will call these the *dynamic types* of buyers and sellers. The market distributions of $\tilde{v}(v)$ and $\tilde{c}(c)$ are denoted as $\tilde{\Phi}$ and $\tilde{\Gamma}$.

In each meeting, for given types $v, c \in [0,1]$, the ex post budget balanced DBM induces the trading probability $q(v,c)$ and the expected payment $t(v,c)$ made by the buyer to the seller. Given that the partner types are drawn from the market distributions Φ and Γ, the associated interim probabilities and payments are

$$q_B(v) = \int q(v,c)\, d\Gamma(c), \quad t_B(v) = \int t(v,c)\, d\Gamma(c)$$

for buyers, and

$$q_S(c) = \int q(v,c)\, d\Phi(v), \quad t_S(c) = \int t(v,c)\, d\Phi(v)$$

for sellers. We invoke the revelation principle and assume that the DBM satisfies IC and IR. Let

$$u_B(v,v') = \tilde{v}(v)\, q_B(v') - t_B(v'),$$
$$u_S(c,c') = t_S(c') - \tilde{c}(c)\, q_S(c')$$

be the interim DBM utilities for v type buyers and c type sellers, over and above their market search values, if they report v' and c'. The IC conditions here mean that these utilities are maximal under truthful reporting:

$$U_B(v) \equiv u_B(v,v) = \max_{v' \in [0,1]} u_B(v,v'), \tag{2}$$

$$U_S(c) \equiv u_S(c,c) = \max_{c' \in [0,1]} u_S(c,c'). \tag{3}$$

The IR condition means that $U_B(v) \geq 0$ and $U_S(c) \geq 0$; however, we restrict attention to ex post IR mechanisms, i.e. for any (v,c),

$$\tilde{c}(c)\, q(v,c) \leq t(v,c) \leq \tilde{v}(v)\, q(v,c). \tag{4}$$

The market utilities $W_B(v)$ and $W_S(c)$ must satisfy the recursive equations

$$W_B(v) = \ell_B U_B(v) + R_\tau W_B(v) - \tau\kappa_B, \tag{5}$$

$$W_S(c) = \ell_S U_S(c) + R_\tau W_S(c) - \tau\kappa_S. \tag{6}$$

Note that $W_B(v)$ and $W_S(c)$ are defined for all types $v, c \in [0,1]$, not only those that are active.

The steady-state assumption implies the following mass balance conditions for $v \in A_B$, $c \in A_S$:

$$b \cdot dF(v) = B\ell_B(\zeta) q_B(v) \cdot d\Phi(v), \quad s \cdot dG(c) = S\ell_S(\zeta)\, q_S(c) \cdot d\Gamma(c). \tag{7}$$

These equations complete the description of the market equilibrium $E = (q_B, q_S, t_B, t_S, W_B, W_S, B, S, \Phi, \Gamma)$.

3 Basic Properties of Equilibria

Our first result gives basic equilibrium properties for any IC and IR DBM.

Lemma 1 *The sets of active trader types are intervals:* $A_B = [\underline{v}, 1]$ *and* $A_S = [0, \bar{c}]$. *The trading probability* $q_B(v)$ *is strictly positive and nondecreasing in* v *on* A_B*, while* $q_S(c)$ *is strictly positive and nonincreasing in* c *on* A_S*. Moreover,*

$$W_B(v) = \int_{\underline{v}}^{v} \frac{\ell_B q_B(x)}{1 - R_\tau + R_\tau \ell_B q_B(x)} dx \quad \textit{for all } v \in [\underline{v}, 1], \tag{8}$$

$$W_S(c) = \int_{c}^{\bar{c}} \frac{\ell_S q_S(x)}{1 - R_\tau + R_\tau \ell_S q_S(x)} dx \quad \textit{for all } c \in [0, \bar{c}]. \tag{9}$$

The functions $\tilde{v}$ *and* $\tilde{c}$ *are absolutely continuous and nondecreasing. Their slopes are*

$$\tilde{v}'(v) = \frac{1 - R_\tau}{1 - R_\tau + R_\tau \ell_B q_B(v)} \quad (a.e.\ v \in A_B), \tag{10}$$

$$\tilde{c}'(c) = \frac{1 - R_\tau}{1 - R_\tau + R_\tau \ell_S q_S(c)} \quad (a.e.\ c \in A_S). \tag{11}$$

This lemma is proved in the Appendix. To gain the intuition for e.g. (8), assume that W_B is differentiable on A_B. Then the Envelope Theorem applied to the IC condition (2) yields for any $v \in A_B$,

$$\begin{aligned} U_B'(v) &= \tilde{v}'(v)\, q_B(v) \\ &= \big(1 - R_\tau W_B'(v)\big)\, q_B(v). \end{aligned} \tag{12}$$

Differentiating the recursive equation (5) and substituting the slope $U_B'(v)$ from (12), we have

$$\begin{aligned} W_B'(v) &= \ell_B U_B'(v) + R_\tau W_B'(v) \\ &= \ell_B \big(1 - R_\tau W_B'(v)\big)\, q_B(v) + R_\tau W_B'(v) \end{aligned}$$

for $v \in A_B$. Solving the above equation for $W_B'(v)$ yields the integrand that appears in (8).

Since the sets of active trader types are intervals, we call $\underline{v}$ and $\bar{c}$ the *marginal participating types*, or *marginal entrants*. Since $W_B(\underline{v}) = W_S(\bar{c}) = 0$, the marginal participating types are equal to the corresponding dynamic types: $\bar{c} = \tilde{c}(\bar{c})$, $\underline{v} = \tilde{v}(\underline{v})$.

Evaluating (5) and (6) at $v = \underline{v}$ and $c = \bar{c}$, we obtain

$$U_B(\underline{v}) = \frac{\tau \kappa_B}{\ell_B(\zeta)}, \tag{13}$$

$$U_S(\bar{c}) = \frac{\tau \kappa_S}{\ell_S(\zeta)}. \tag{14}$$

In other words, the expected mechanism payoffs for the marginal buyers and sellers are just sufficient to cover their expected search costs until the next meeting.

Define

$$\underline{c} \equiv \tilde{c}(0), \quad \bar{v} \equiv \tilde{v}(1).$$

If the DBM is ex post IR, then necessarily

$$\underline{c} < \underline{v}, \quad \bar{c} < \bar{v}. \tag{15}$$

Otherwise say the marginal buyers would not be able to trade profitably with even the lowest cost sellers, because the latter would prefer to search for a better match in the market.[12]

We define the competitive, or Walrasian, price p^* as the price that clears the flows of the arriving cohorts:

$$b[1 - F(p^*)] = sG(p^*).$$

From the steady-state mass balance condition $b[1 - F(\underline{v})] = sG(\bar{c})$, the marginal participating types $\underline{v}$ and $\bar{c}$ must be on different sides of the Walrasian price p^*, i.e. either $\bar{c} \le p^* \le \underline{v}$ or $\underline{v} \le p^* \le \bar{c}$. Therefore (15) implies that p^* must always fall within the acceptance interval, i.e. $p^* \in [\underline{c}, \bar{v}]$.

4 NBP Mechanisms and their Rate of Convergence

We now introduce our nonvanishing bargaining power (NBP) conditions that will imply convergence to perfect competition. Let

$$U(v, c) = v - c - R_\tau (W_B(v) + W_S(c))$$

be the bargaining surplus available to share in a given meeting of a v-type buyer and c-type seller. Our conditions generalize the essential property of the random proposer take it or leave it games, namely that each trader has some bargaining power, in the sense that he or she can guarantee some fraction of the bargaining surplus. From now on, we will often use the notation U_τ, $U_{B\tau}$, $U_{S\tau}$, etc. to emphasize the dependence of equilibrium objects on τ. Our NBP conditions require that, for any active buyer of type v, his DBM utility $U_{B\tau}(v)$ is at least some fixed, independent of τ, fraction of the *minimal* bargaining surplus $\min_{c \in A_{S\tau}} U_\tau(v, c)$ in his meetings in the market, if there is any. Similarly, any active seller of type c can guarantee a certain fraction of the minimal bargaining surplus $\min_{v \in A_{B\tau}} U_\tau(v, c)$.

Assumption 2 (Buyer NBP condition) *There exists $\alpha_B > 0$ such that $\forall v \in A_{B\tau}$ and $\forall \tau > 0$ sufficiently small,*

$$U_{B\tau}(v) \ge \alpha_B \min_{c \in A_{S\tau}} U_\tau(v, c). \tag{16}$$

[12] For example, to prove $\underline{c} < \underline{v}$, suppose $\underline{v} \le \underline{c}$. Then for any $c \in A_S$, the ex post IR property (4) implies

$$\underline{v} q(\underline{v}, c) - t(\underline{v}, c) \le [\underline{v} - \tilde{c}(c)] q(\underline{v}, c) \le (\underline{v} - \underline{c}) q(\underline{v}, c) \le 0.$$

But it implies $U_B(\underline{v}) = 0$, contradicting (13).

Assumption 3 (Seller NBP condition) *There exists* $\alpha_S > 0$ *such that* $\forall c \in A_{S\tau}$ *and* $\forall \tau > 0$ *sufficiently small,*

$$U_{S\tau}(c) \geq \alpha_S \min_{v \in A_{B\tau}} U_\tau(v, c) . \tag{17}$$

Definition 1 *A DBM that satisfies IC, ex post IR, and the two NBP conditions above is called an NBP bargaining mechanism.*

Remark 1 *Since the dynamic type functions* $\tilde{v}_\tau, \tilde{c}_\tau$ *are nondecreasing,* $\tilde{v}_\tau(v) \geq \underline{v}_\tau$ *and* $\tilde{c}_\tau(c) \leq \bar{c}_\tau$*. Moreover,*

$$U_\tau(v, c) = \tilde{v}_\tau(v) - \tilde{c}_\tau(c) .$$

It follows that the NBP conditions can be equivalently stated as

$$U_{B\tau}(v) \geq \alpha_B (\tilde{v}_\tau(v) - \bar{c}_\tau), \quad U_{S\tau}(c) \geq \alpha_S (\underline{v}_\tau - \tilde{c}_\tau(c)) . \tag{18}$$

The maximum DBM surplus ever available for buyers and sellers in the market equilibrium is

$$\max_{v \in A_{B\tau}, c \in A_{S\tau}} U_\tau(v, c) = \bar{v}_\tau - \underline{c}_\tau .$$

Our main result in this section is the following theorem and its corollary. The theorem provides a lower and upper bounds on $\bar{v}_\tau - \underline{c}_\tau$, and the corollary provides an upper bound on the deviations of the equilibrium utilities $W_{B\tau}(v)$ and $W_{S\tau}(c)$ from the Walrasian utilities. Before stating the theorem, we define what it means for a sequence to converge at a linear rate.

Definition 2 *For any real sequence* $\{x_\tau\}$ *indexed by* $\tau > 0$ *with* $\tau \to 0$*, we say that* $x_\tau \to 0$ *as* $\tau \to 0$ *at a linear rate if and only if* $\exists \underline{x}, \bar{x} \in \mathbb{R}_{++}$ *such that for all sufficiently small* $\tau > 0$, $\tau \cdot \underline{x} \leq |x_\tau| \leq \tau \cdot \bar{x}$.[13]

Theorem 1 *For any sequence of market equilibria* E_τ *with* $\tau \to 0$,

$$\max_{v \in A_{B\tau}, c \in A_{S\tau}} U_\tau(v, c) \to 0$$

as $\tau \to 0$ *at a linear rate if and only if the NBP conditions* (16) *and* (17) *hold. Moreover, under these conditions,*

$$\tau \cdot \kappa \leq \bar{v}_\tau - \underline{c}_\tau \leq \tau \cdot \frac{K(\zeta_0)}{\alpha_B + \alpha_S} \left(1 + \frac{2r}{\kappa}\right)^3 . \tag{19}$$

where $\kappa \equiv \min\{\kappa_B, \kappa_S\}$ *and*

$$\zeta_0 \equiv \frac{\alpha_B}{\alpha_S} \frac{\kappa_S}{\kappa_B} .$$

Note that the bounds in Theorem 1 do not depend on the distributions F and G.

The proof of Theorem 1 relies on the following lemma showing that (16) and (17) imply that the entry gap $\underline{v}_\tau - \bar{c}_\tau$, if any, is bounded by $K(\zeta_0) / (\alpha_B + \alpha_S)$.

[13] Equivalently, $\lim_{\tau \to 0} \sup |x_\tau| / \tau < \infty$ and $\lim_{\tau \to 0} \inf |x_\tau| / \tau > 0$.

Lemma 2 (Bound for entry gap) *For any NBP bargaining mechanism, we have*

$$\max\{\underline{v}_\tau - \bar{c}_\tau, 0\} \le \tau \cdot \min\left\{\frac{\kappa_B}{\alpha_B \ell_B(\zeta)}, \frac{\kappa_S}{\alpha_S \ell_S(\zeta)}\right\} \le \tau \cdot \frac{K(\zeta_0)}{\alpha_B + \alpha_S}. \tag{20}$$

Lemma 2 implies that, if there is a positive entry gap $\underline{v}_\tau - \bar{c}_\tau$, it is $O(\tau)$. For the length of the acceptance interval $[\underline{c}_\tau, \bar{v}_\tau]$ we can write:

$$\begin{aligned}\bar{v}_\tau - \underline{c}_\tau &\le (\bar{c}_\tau - \underline{c}_\tau) + (\underline{v}_\tau - \bar{c}_\tau) + (\bar{v}_\tau - \underline{v}_\tau)\\ &= \int_0^{\bar{c}_\tau} \tilde{c}'_\tau(c)dc + (\underline{v}_\tau - \bar{c}_\tau) + \int_{\underline{v}_\tau}^1 \tilde{v}'_\tau(v)dv.\end{aligned}$$

The idea of the proof of the sufficiency result in Theorem 1 is to bound the slopes of dynamic types $\tilde{v}'_\tau(v)$ and $\tilde{c}'_\tau(c)$. Recall Lemma 1; it implies for a.e. active buyers and sellers

$$\tilde{v}'_\tau(v) = \frac{\tau r}{\ell_B(\zeta_\tau) q_{B\tau}(v)} + o(\tau), \quad \tilde{c}'_\tau(c) = \frac{\tau r}{\ell_S(\zeta_\tau) q_{S\tau}(c)} + o(\tau).$$

In the proof, we show that (a) ζ_τ is uniformly bounded from above and below, so that both $\ell_B(\zeta_\tau)$ and $\ell_S(\zeta_\tau)$ are bounded from below for small $\tau > 0$, and (b) the probabilities of trading $q_{B\tau}(v)$ and $q_{S\tau}(c)$ are also bounded from below uniformly for all active types. Then it is clear that the slopes $\tilde{v}'_\tau(v)$ and $\tilde{c}'_\tau(c)$ will converge to 0 at a linear rate. The intuition for the lower bound in (19) is that, for a given ζ, ex post IR implies that the expected profit of type $\bar{v}_\tau$ buyer in a given meeting is no more than $\ell_B(\zeta_0)(\bar{v}_\tau - \underline{c}_\tau)$ and therefore $\bar{v}_\tau - \underline{c}_\tau \ge \tau\kappa_B/\ell_B(\zeta_\tau) \ge \tau\kappa_B$. Similarly, $\bar{v}_\tau - \underline{c}_\tau \ge \tau\kappa_S$.

Define traders' Walrasian utilities in the usual manner, as

$$W^*_B(v) = \max\{v - p^*, 0\}, \qquad W^*_S(c) = \max\{p^* - c, 0\}.$$

Let

$$W^+_{B\tau}(v) = \max\{W_{B\tau}(v), 0\}, \quad W^+_{S\tau}(c) = \max\{W_{S\tau}(c), 0\}$$

be the traders' equilibrium market utilities.

Corollary 1 *For any sequence of market equilibria E_τ with $\tau \to 0$, if the NBP conditions* (16) *and* (17) *hold, then $\forall v, c \in [0,1]$,*

$$\max\left\{|W^*_B(v) - W^+_{B\tau}(v)|, |W^*_S(c) - W^+_{S\tau}(c)|\right\} \le \tau \cdot \left[r + \frac{K(\zeta_0)}{\alpha_B + \alpha_S}\left(1 + \frac{2r}{\kappa}\right)^3\right]. \tag{21}$$

In particular, as $\tau \to 0$, the rate of convergence of equilibrium utilities $W^+_{B\tau}(v)$ and $W^+_{S\tau}(c)$ is $O(\tau)$.

Remark 2 *Since $W^*_B(v) - W^+_{B\tau}(v)$ and $W^*_S(c) - W^+_{S\tau}(c)$ are not guaranteed to be positive, the absolute values are needed. Indeed, if $\underline{v}_\tau < p^*$, then buyers with type $v \in (\underline{v}_\tau, p^*]$ would have strictly positive utilities in equilibrium but have* 0 *Walrasian utilities. We also do not have a positive lower bound in Corollary 1. Indeed, for some types v, $c \in [0,1]$ we could have $W^*_B(v) = W^+_{B\tau}(v) = 0$ and/or $W^*_S(c) = W^+_{S\tau}(c) = 0$.*

Remark 3 *Our explicit bound* (21) *in Corollary 1 can be tight when frictions are small, even when* τ *does not go to* 0*. For example, suppose that* $\tau = 1$ *and consider the case of small frictions* κ_B*,* κ_S *and* r*. For simplicity, suppose that frictions are proportionally small in the following sense:* $(\kappa_B, \kappa_S, r) = \delta(1, 1, r_0)$*, where* $\delta > 0$ *is small. Then* $r/\kappa = r_0$ *and*

$$K(\zeta_0) = \delta \cdot \left(\frac{1}{\ell_B(\zeta_0)} + \frac{1}{\ell_S(\zeta_0)} \right),$$

where $\zeta_0 = \alpha_B/\alpha_S$*. Clearly, the r.h.s. of* (21) *becomes small with* δ*.*

We now turn to some concrete examples of bargaining protocols whose associated DBMs satisfy Assumptions 2 and 3 in Perfect Bayesian equilibrium.[14]. We assume that bargaining is instantaneous and the protocol does not change with τ.[15] For each bargaining protocol, we verify the NBP conditions by proposing a feasible strategy that guarantees the buyer and seller the payoff bounds in (18).

Example 1 *Random-proposer take it or leave it offer games where either a buyer or a seller is chosen to be a proposer with positive probabilities,* $\alpha_B \in (0, 1)$ *for buyers and* $\alpha_S = 1 - \alpha_B$ *for sellers.* When chosen as the proposer, a buyer has the option to propose $\bar{c}$, and a seller has the option to propose $\underline{v}$. These offers would be accepted in a Perfect Bayesian equilibrium by sellers with $c < \bar{c}$ and buyers with $v > \underline{v}$ respectively. We therefore have the NBP conditions (18).

Example 2 *Repeated take it or leave it offer games in which the seller is chosen as a proposer with probability* $\alpha \in (0, 1)$*, and the buyer is chosen with the complementary probability* $1 - \alpha$*. The proposer then makes* N *sequential offers to the responder. After each rejected offer, the game is exogenously terminated with probability* $\beta \in (0, 1)$*.* In any Perfect Bayesian equilibrium, every trader has a deviation strategy in which he or she would make unacceptable offers until the final round. In the final round, he or she would make an offer that would be surely accepted. For example, a buyer can follow this strategy and guarantee that, ex ante, he is a take it or leave it proposer with probability $\alpha_B = (1 - \alpha)(1 - \beta)^{N-1}$. When he is a take it or leave it proposer, he can deviate and offer the price $p = \bar{c}$. In a Perfect Bayesian equilibrium, this offer would be accepted by any seller having $c < \bar{c}$. A parallel construction for the seller leads to the NBP conditions (18).

Example 3 *Alternate take it or leave it offer games in which the first proposer is chosen to be the seller with probability* $\alpha \in (0, 1)$ *and the buyer with probability* $1 - \alpha$*. The traders alternate making proposals for* N *rounds. Once again, the game may be terminated with probability* $\beta \in (0, 1)$ *after each rejected offer.* Here, in any Perfect Bayesian equilibrium, every trader has a deviation strategy in which he or she would make unacceptable offers and reject all offers until the final round. If he or she is a take it or leave it proposer in the final round, he or she would make an offer that would be surely accepted. For example,

[14] These examples are discussed in e.g. Ausubel, Cramton, and Deneckere (2002).

[15] Our assumption of instantaneous bargaining is natural because we focus on the frictions of costly, time consuming search.

a buyer can follow this strategy and guarantee that, ex ante, he is a take it or leave it proposer with probability $\alpha_B = (1-\alpha)(1-\beta)^{N-1}$ when N is odd and $\alpha_B = \alpha(1-\beta)^{N-1}$ when N is even. When he is a take it or leave it proposer, he can deviate and offer the price $p = \bar{c}$. In a Perfect Bayesian equilibrium, this offer would be accepted by any seller having $c < \bar{c}$. Once again, repeating the logic of Example 1, we can verify the NBP conditions (18).

One can extend the logic of the above examples to show that Assumptions 2 and 3 are satisfied by any instantaneous protocol such that (i) the buyer has a strategy such that the play of the bargaining game passes through a node where he is a take it or leave it proposer with probability at least $\alpha_B > 0$, and (ii) the seller has a strategy such that the play passes through a node where she is a take it or leave it proposer with probability at least $\alpha_S > 0$.[16]

5 Asymptotic Optimality of NBP Mechanisms

Let

$$W^{0*} = b\int_{p^*}^{1}(v-p^*)\,dF(v) + s\int_{0}^{p^*}(p^*-c)\,dG(c).$$

be the Walrasian welfare of a cohort, and let

$$W^0_\tau = b\int W^+_{B\tau}(v)\,dF(v) + s\int W^+_{S\tau}dG(c).$$

be its market welfare. Our main result in this section is that no bargaining mechanism can attain a faster than linear rate of convergence.

Theorem 2 *For any sequence of market equilibria E_τ,*

$$W^{0*} - W^0_\tau \geq \tau \cdot b\left[1 - F(\underline{v}_\tau)\right]\min_{\zeta>0} K(\zeta). \tag{22}$$

As $\tau \to 0$, the rate of convergence of $W^{0*} - W^0_\tau$ to 0 cannot be faster than linear.

Corollary 2 *Any random-proposer take it leave it offer protocol is asymptotically optimal, in the sense that $W^0_\tau \to W^{0*}$ at a linear rate.*

Remark 4 *The proof of Theorem 2 only requires the individual rationality of the bargaining mechanism and therefore also applies to the full information setting as in e.g. Mortensen and Wright (2002).*

The intuition for Theorem 2 is that with matching frictions, the loss of welfare due to costly delay is unavoidable regardless of the bargaining protocol. An interesting further question is how to isolate the welfare loss due to strategic behavior. One way of doing so is to compensate the traders for the costs of participation and time discounting that they

[16]The previous version of this paper, available at `http://artyom239.googlepages.com/DMBG_rate_of_convergence_discrete_Ju.pdf`, called these protocols *generalized random proposer TIOLI*, *(GRP TIOLI)*.

incur in equilibrium. To formalize the idea, one can eliminate $W_{B\tau}(v)$ for $v \in A_{B\tau}$ and $W_{S\tau}(c)$ for $c \in A_{S\tau}$ from the recursive equations (5) and (6):

$$W_{B\tau}(v) = \frac{\ell_B \left[q_{B\tau}(v)\, v - t_{B\tau}(v)\right] - \tau\kappa_B}{1 - R_\tau + R_\tau \ell_B q_{B\tau}(v)}, \quad W_{S\tau}(c) = \frac{\ell_S \left[t_{S\tau}(c) - q_{S\tau}(c)\, c\right] - \tau\kappa_S}{1 - R_\tau + R_\tau \ell_S q_{S\tau}(c)}. \quad (23)$$

Eliminating the time discounting and participation costs from the value functions in (23), we can define the welfare loss due to strategic effects as

$$\Delta w_\tau = W^{0*} - b \int_{A_{B\tau}} \left(v - \frac{t_{B\tau}(v)}{q_{B\tau}(v)} \right) dF(v) - s \int_{A_{S\tau}} \left(\frac{t_{S\tau}(c)}{q_{S\tau}(c)} - c \right) dG(c) .$$

Since $bdF(v)/q_{B\tau}(v) = B_\tau \ell_B d\Phi_\tau(v)$, $sdG(c)/q_{S\tau}(c) = S_\tau \ell_S d\Gamma_\tau(c)$, and the transfers are balanced,

$$B_\tau \ell_B \int_{v \in A_{B\tau}} t_{B\tau}(v)\, d\Phi_\tau(v) = S_\tau \ell_S \int_{c \in A_{S\tau}} t_{S\tau}(c)\, d\Gamma_\tau(c) ,$$

we have

$$\Delta w_\tau = W^{0*} - b \int_{A_{B\tau}} v dF(v) + s \int_{A_{S\tau}} c dG(c) .$$

By Lemma 1, the sets of participating types $A_{B\tau}$ and $A_{S\tau}$ are intervals, i.e. $A_{B\tau} = [\underline{v}_\tau, 1]$ and $A_{S\tau} = [0, \bar{c}_\tau]$. The welfare loss Δw_τ is then equal to the area of the familiar "deadweight loss" triangle,

$$\Delta w_\tau = b \int_{p^*}^{\underline{v}_\tau} (v - p^*)\, dF(v) + s \int_{\bar{c}_\tau}^{p^*} (p^* - c)\, dG(c) .$$

The inefficiency loss Δw_τ is due only to inefficient entry, because the marginal participating types $\underline{v}_\tau$, $\bar{c}_\tau$ may be different from p^*. As is well known, the area of the "deadweight loss" triangle is asymptotically proportional to $|\underline{v}_\tau - \bar{c}_\tau|^2$, and therefore $\Delta w_\tau = O\left(|\underline{v}_\tau - \bar{c}_\tau|^2\right)$. Theorem 1 implies the following corollary.

Corollary 3 *For any sequence of market equilibria E_τ satisfying the NBP conditions, the welfare loss due to strategic behavior $\Delta w_\tau = O\left(\tau^2\right)$ converges to 0 at a no slower than quadratic rate.*

6 Nonconvergent and Slow Convergent Equilibria of the k-Double Auction

Recall the rules of the bilateral k-double auction: the buyer and the seller simultaneously and independently submit a bid price p_B and an ask price p_S respectively, and then trade occurs if and only if the buyer's bid is at least as high as the seller's ask, at the weighted average price $(1-k)p_S + kp_B$, where $k \in (0,1)$. In this section we show that the dynamic matching market with the bilateral k-double auction has sequences of equilibria that do not converge to perfect competition at a linear rate. Moreover, we show that some sequences of equilibria do not converge at all.

We distinguish two classes of double-auction equilibria: full trade and nonfull trade. A full trade equilibrium is characterized by the property that every meeting results in trade. We claim that the class of full trade equilibria includes equilibria that are very inefficient, even with arbitrarily small frictions. (But it also includes equilibria that converge to perfect competition.)

From Lemma 1, the sets A_B, A_S of active types are still intervals $[\underline{v}, 1]$ and $[0, \bar{c}]$ for some marginal types $\underline{v}$ and $\bar{c}$; and we also have $\tilde{v}(v) < v$ and $\tilde{c}(c) > c$ for all $v > \underline{v}$ and all $c < \bar{c}$. Since all active traders' trading probabilities are strictly positive, they must in equilibrium submit serious bids/asks, and therefore, we must have $p_B(v) \leq \tilde{v}(v) < v$ and $p_S(c) \geq \tilde{c}(c) > c$ for all $v > \underline{v}$ and all $c < \bar{c}$. Now it is clear that for an equilibrium to be full trade, we must have $\bar{c} \leq \underline{v}$, and all traders must submit a common bid/ask p. Hence every matched pair trades at the price p. Furthermore, $\underline{v}$-buyers and $\bar{c}$-sellers have to recover their participation costs, thus in any full trade equilibrium we have $\bar{c} < p < \underline{v}$ for some $p \in (0, 1)$.[17]

Any full trade equilibrium must satisfy indifference equations for the marginal types, as well as the mass balance equation:

$$\ell_B(\zeta)(\underline{v} - p) = \tau\kappa_B, \tag{24}$$

$$\ell_S(\zeta)(p - \bar{c}) = \tau\kappa_S, \tag{25}$$

$$b[1 - F(\underline{v})] = sG(\bar{c}). \tag{26}$$

The converse is also true, i.e. any quadruple $\{p, \zeta, \underline{v}, \bar{c}\}$ satisfying (24), (25), (26) and $\tau \cdot K(\zeta) < 1$ must characterize a full trade equilibrium. In particular, any trader's best-response bid/ask strategy is p, given that all other active traders use this strategy.

From equations (24) and (25), it follows that the entry gap is

$$\underline{v} - \bar{c} = \tau K(\zeta). \tag{27}$$

The next proposition shows that $\underline{v} - \bar{c}$ can be arbitrarily close to 1 for all τ small enough so that a non-trivial equilibrium exists. Therefore equilibrium outcomes can be arbitrarily far from efficiency even with small frictions. The set of equilibrium entry gaps converges to $(0, 1)$ as frictions disappear, so the set of full trade equilibria ranges from perfectly competitive to almost perfectly inefficient. Moreover, the set of equilibrium prices also converges to $(0, 1)$. Thus indeterminacy grows rather than vanishes with competition, contrary to the results in the static double auction literature.

Proposition 1 *Under bilateral k-double auction, a full trade equilibrium exists if and only if*

$$\tau \cdot \min_{\zeta > 0} K(\zeta) < 1. \tag{28}$$

The set of equilibrium values of $\underline{v} - \bar{c}$ is an interval $[\tau \cdot \min_{\zeta>0} K(\zeta), 1)$. As $\tau \to 0$, this set and the set of equilibrium prices converge to $(0, 1)$ In particular, there exists a sequence of full trade equilibria that converges to perfect competition, but also sequences that do not converge.

[17] It has been known at least since Chatterjee and Samuelson (1983) and Leininger, Linhart, and Radner (1989) that k-DAs have multiple equilibria. The latter paper in fact looks at equilibria where buyers and sellers can trade only at price p, and shows that such equilibria exist for any price $p \in (0, 1)$.

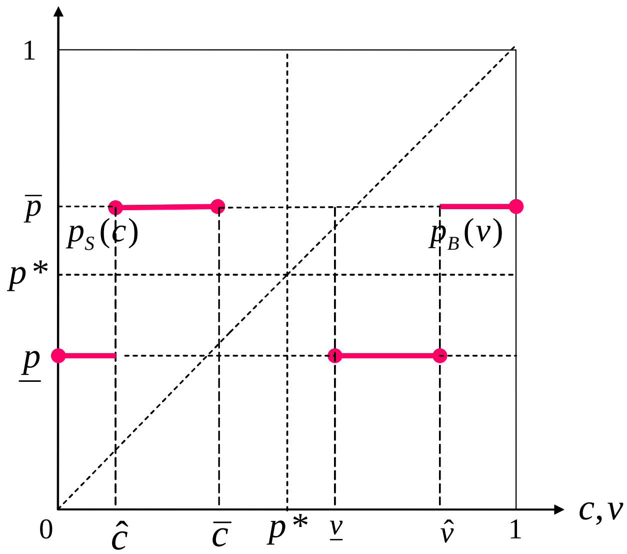

Figure 1: A two-step equilibrium of the k-double auction

By looking at the equilibria with the smallest entry gap, $\underline{v} - \bar{c} = \tau \cdot \min_{\zeta>0} K(\zeta)$, we can see that the following corollary is true.

Corollary 4 *There are full trade equilibria that converge, in terms of the ex ante utilities, at a linear rate.*

It is not hard to see that the condition $\tau \cdot \min_{\zeta>0} K(\zeta) < 1$ is also necessary for any nontrivial steady-state equilibrium to exist. We thus have the following corollary.

Corollary 5 *There exists a nontrivial steady-state equilibrium (either full trade or nonfull trade) if and only if* $\tau \cdot \min_{\zeta>0} K(\zeta) < 1$.

Remark 5 *This necessary and sufficient condition is weaker than the one for the random-proposer take it or leave it offer bargaining, as shown in Shneyerov and Wong (forthcoming).*

Note that the sequences of k-DA equilibria in Proposition 1 have the property that market becomes extremely unbalanced as $\tau \to 0$: the buyer to seller ratio ζ_τ either tends to 0 (if $p > p^*$) of to ∞ (if $p < p^*$). We now proceed to show that there also exist nonfull trade k-DA equilibria that do not converge to perfect competition, even though ζ_τ remains bounded from above and below along the sequence. In the theorem below, we show existence of equilibria with two steps (see Figure 1). There are two seller cutoff types $\hat{c} \in (0,1)$ and $\bar{c} \in (0,1)$ with $\hat{c} < \bar{c}$, and two buyer cutoff types $\hat{v} \in (0,1)$ and $\underline{v} \in (0,1)$ with $\hat{v} > \underline{v}$. The sellers with $c \in [0, \hat{c})$ enter and bid $p_S(c) = \underline{p}$, where $\underline{p}$ is some constant strictly below p^*. The sellers with $c \in [\hat{c}, \bar{c}]$ enter and bid $p_S(c) = \bar{p}$, where $\bar{p} > p^*$. The sellers with $c \in (\bar{c}, 1]$ do not enter. Similarly, the buyers with $v \in (\hat{v}, 1]$ enter and bid $\bar{p}$, the buyers with $v \in [\underline{v}, \hat{v}]$ enter and submit $\underline{p}$, and the buyers with $v \in [0, \underline{v})$ do not enter.

The following theorem contains our non-convergence result for the two-step class of equilibria.

Theorem 3 *For any $a \in (0,1)$, there exist $r_0 > 0$, $\tau_0 > 0$ and $\bar{W} < W^{0*}$ such that for all $r \in [0, r_0)$ and $\tau \in (0, \tau_0)$, there exists a two-step equilibrium in which the price spread is larger than a, i.e. $\bar{p} - \underline{p} > a$, and the total ex ante surplus is smaller than $\bar{W}$, i.e. $W^0 < \bar{W}$.*

In both examples (full and nonfull trade), the existence of slow (or non) convergent sequences of equilibria can be traced to the violation of the NBP conditions (16) and (17). The marginal types $v = \underline{v}_\tau$ and $c = \bar{c}_\tau$ have interim utilities $U_{B\tau}(\underline{v}_\tau)$ and $U_{S\tau}(\bar{c}_\tau)$ just sufficient to cover their expected search costs until the next meeting (see (13) and (14)):

$$U_{B\tau}(\underline{v}_\tau) = \frac{\tau \kappa_B}{\ell_B(\zeta_\tau)}, \quad U_{S\tau}(\bar{c}_\tau) = \frac{\tau \kappa_S}{\ell_S(\zeta_\tau)}.$$

For the marginal types, the NBP conditions (see Remark 1) require existence of $\alpha_B > 0$ and $\alpha_S > 0$ such that $\forall \tau > 0$ sufficiently small,

$$\frac{\kappa_B}{\ell_B(\zeta_\tau)} \geq \alpha_B \frac{\underline{v}_\tau - \bar{c}_\tau}{\tau}, \tag{29}$$

$$\frac{\tau \kappa_S}{\ell_S(\zeta_\tau)} \geq \alpha_S \frac{\underline{v}_\tau - \bar{c}_\tau}{\tau}. \tag{30}$$

Consider full trade equilibria that are slowly (or non) convergent: let $(\underline{v}_\tau - \bar{c}_\tau)/\tau \to \infty$. Obviously $\ell_B(\zeta_\tau)$ and $\ell_S(\zeta_\tau)$ cannot be both convergent to 0. If $\ell_B(\zeta_\tau)$ is not convergent to 0, then (29) is violated, while if $\ell_S(\zeta_\tau)$ is not convergent to 0, then (30) is violated.

Unlike in Proposition 1, the construction in the proof of Theorem 3 treats buyers and sellers symmetrically. In particular, ζ_τ could be fixed at any value, say $\zeta_\tau = \zeta_0$. Now consider nonfull trade equilibria that are slow (or non) convergent and have $\zeta_\tau = \zeta_0$ for all sufficiently small $\tau > 0$. In these equilibria, *both* (29) and (30) are necessarily violated as $\tau \to 0$.

The rules of the double auction do not guarantee that each trader has at least some bargaining power. On the other hand, in the NBP mechanisms, both parties have some bargaining power. This creates strong incentive to enter for both buyers and sellers and drives the marginal participating types close to each other and at the same time close to the Walrasian price.

7 Concluding Remarks

In a framework of dynamic matching and bargaining, we have provided a complete characterization of bilateral trade mechanism sequences that converge to perfect competition at a linear rate, which is shown to be the fastest possible. The conditions are easy to check for a number of interesting bargaining protocols. Any random-proposer take it or leave it offer mechanism satisfies these conditions, while k-double auctions violate them and have equilibria that converge to perfect competition slowly, or not at all.

We believe that the mechanism design approach to bilateral search that we have developed in this paper would be useful for showing convergence to perfect competition (with rate) in other settings such as multilateral bargaining in a market with many-to-many matches as e.g. in Dagan, Serrano, and Volij (2000), or on networks as in Abreu and Manea (2008). We are currently pursuing these extensions.

A different issue is that our non-convergent examples for the k-double auction require a great deal of coordination among the traders. Additional assumptions, e.g. the continuity of strategies, could be imposed to restrict the set of equilibria with the purpose of proving their convergence at a linear rate. In addition, allowing a multilateral matching technology might also restore convergence of all equilibria of the k-double auction mechanism. These extensions are left for future research.

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Appendix

Proof of Lemma 1

We prove the results for buyers only; the argument for the sellers is parallel. We begin by noting that the IC condition (2) and the recursive equation (5) together imply that $\forall v, v' \in [0,1]$,

$$\left[1 - R_\tau + R_\tau \ell_B q_B\left(v'\right)\right] W_B(v) \geq \ell_B \left[v q_B\left(v'\right) - t_B\left(v'\right)\right] - \tau \kappa_B, \tag{31}$$

and the inequality becomes equality if $v' = v$. For $v \in A_B$, the l.h.s. is non-negative, so it follows that $v q_B(v) > t_B(v)$. Then IR condition implies $v > 0$ and $q_B(v) > 0$ for any $v \in A_B$.

Condition (31) implies

$$W_B(v) = \max_{v' \in [0,1]} \frac{\ell_B \left[v q_B(v') - t_B(v')\right] - \tau \kappa_B}{1 - R_\tau + R_\tau \ell_B q_B(v')}.$$

As the maximum of a set of nondecreasing and affine functions, W_B is absolutely continuous and convex, and therefore differentiable almost everywhere on $[0,1]$. The Envelope Theorem (Theorem 2 in Milgrom and Segal (2002)) then implies

$$W_B'(v) = \frac{\ell_B q_B(v)}{1 - R_\tau + R_\tau \ell_B q_B(v)} \text{ a.e. on } [0,1],$$

and (10) must also hold. From the continuity of W_B, we have $W_B(\underline{v}) = 0$ and hence (8) follows. Since $q_B(v) > 0$ for $v \in A_B$, W_B is increasing on A_B, and therefore A_B must be an interval $[\underline{v}, 1]$. The convexity of W_B implies that q_B is nondecreasing. Q.E.D.

Proof of Lemma 2

By the buyer NBP condition, $U_{B\tau}(\underline{v}_\tau) \geq \alpha_B(\underline{v}_\tau - \bar{c}_\tau)$. The equilibrium condition (13) then implies $\kappa_B \tau \geq \alpha_B \ell_B(\zeta)(\underline{v}_\tau - \bar{c}_\tau)$. Similarly, we can show that $\kappa_S \tau \geq \alpha_S \ell_S(\zeta)(\underline{v}_\tau - \bar{c}_\tau)$, and therefore

$$\underline{v}_\tau - \bar{c}_\tau \leq \tau \cdot \min\left\{\frac{\kappa_B}{\alpha_B \ell_B(\zeta)}, \frac{\kappa_S}{\alpha_S \ell_S(\zeta)}\right\}.$$

Now since ℓ_B is nonincreasing and ℓ_S is nondecreasing, the above minimum is maximized at $\zeta = \zeta_0$ because

$$\frac{\kappa_B}{\alpha_B \ell_B(\zeta_0)} = \frac{\kappa_S}{\alpha_S \ell_S(\zeta_0)},$$

and the resulting value of the maximum is $K(\zeta_0)/(\alpha_B + \alpha_S)$. Q.E.D.

Proof of Theorem 1

Sufficiency proof:

Step 1: We claim that

$$\text{(a): } \frac{\underline{v}_\tau - \underline{c}_\tau}{\bar{v}_\tau - \underline{c}_\tau} \geq \frac{\kappa_B}{r + \kappa_B}$$

$$\text{(b):}\quad \frac{\bar{v}_\tau - \bar{c}_\tau}{\bar{v}_\tau - \underline{c}_\tau} \geq \frac{\kappa_S}{r+\kappa_S}.$$

We provide the proof for part (a) only. The proof for part (b) is the flip of that for part (a). First note that the ex post IR condition (4) implies $t_{B\tau}(\underline{v}_\tau) \geq \underline{c}_\tau q_{B\tau}(\underline{v}_\tau)$. Otherwise there is some $c \in A_{S\tau}$ such that $t_\tau(\underline{v}_\tau, c) < \underline{c}_\tau q_\tau(\underline{v}_\tau, c) \leq \tilde{c}_\tau(c)\, q_\tau(\underline{v}_\tau, c)$, contradicting (4).

Since $q_{B\tau}$ is nondecreasing, (13) then implies that for any $v \in [\underline{v}_\tau, 1]$,

$$\ell_B q_{B\tau}(v)(\underline{v}_\tau - \underline{c}_\tau) \geq \ell_B q_{B\tau}(\underline{v}_\tau)(\underline{v}_\tau - \underline{c}_\tau) \geq \ell_B \left[q_{B\tau}(\underline{v}_\tau)\underline{v}_\tau - t_{B\tau}(\underline{v}_\tau)\right] = \kappa_B \tau,$$

and therefore

$$\ell_B q_{B\tau}(v) \geq \frac{\kappa_B \tau}{\underline{v}_\tau - \underline{c}_\tau}. \tag{32}$$

Then for almost all $v \in [\underline{v}_\tau, 1]$,

$$\tilde{v}'_\tau(v) = \frac{1-R_\tau}{1-R_\tau + R_\tau \ell_B q_{B\tau}(v)} \leq \frac{r\tau}{\ell_B q_{B\tau}(v)} \leq \frac{r}{\kappa_B/(\underline{v}_\tau - \underline{c}_\tau)},$$

where the second last inequality follows by the concavity of the function $1-e^{-x}$. Hence

$$\bar{v}_\tau - \underline{v}_\tau = \int_{\underline{v}_\tau}^1 \tilde{v}'_\tau(v)dv \leq \frac{r}{\kappa_B/(\underline{v}_\tau - \underline{c}_\tau)},$$

$$\frac{\bar{v}_\tau - \underline{v}_\tau}{\underline{v}_\tau - \underline{c}_\tau} \leq \frac{r}{\kappa_B},$$

$$\frac{\underline{v}_\tau - \underline{c}_\tau}{\bar{v}_\tau - \underline{c}_\tau} = \frac{1}{1 + (\bar{v}_\tau - \underline{v}_\tau)/(\underline{v}_\tau - \underline{c}_\tau)} \geq \frac{1}{1+\frac{r}{\kappa_B}} = \frac{\kappa_B}{r+\kappa_B}.$$

Step 2: We claim that

$$\text{(a):}\quad \min\{\underline{v}_\tau, \bar{c}_\tau\} - \underline{c}_\tau \leq \frac{\tau 4r\,(r+\kappa_B)}{\ell_S \alpha_S \kappa_B}$$

$$\text{(b):}\quad \bar{v}_\tau - \max\{\underline{v}_\tau, \bar{c}_\tau\} \leq \frac{\tau 4r(r+\kappa_S)}{\ell_B \alpha_B \kappa_S}.$$

Again by symmetry, we only provide a proof for (a). Let $y \equiv \min\{\underline{v}_\tau, \bar{c}_\tau\} - \underline{c}_\tau$. Consider a type c seller with $\tilde{c}_\tau(c) \leq \underline{c}_\tau + y/2$. By the seller NBP condition,

$$U_{S\tau}(c) \geq \alpha_S \left[\underline{v}_\tau - \tilde{c}_\tau(c)\right].$$

Since $\underline{v}_\tau - \tilde{c}_\tau(c) \geq \underline{v}_\tau - (\underline{c}_\tau + y/2)$, and our definition of y implies that $y \leq \underline{v}_\tau - \underline{c}_\tau$, it follows that $\underline{v}_\tau - \tilde{c}_\tau(c) \geq (\underline{v}_\tau - \underline{c}_\tau)/2$ and therefore

$$t_{S\tau}(c) - \tilde{c}_\tau(c) q_{S\tau}(c) = U_{S\tau}(c) \geq \alpha_S \frac{\underline{v}_\tau - \underline{c}_\tau}{2}.$$

Moreover, the ex post IR condition (4) implies $t_{S\tau}(c) \leq \bar{v}_\tau q_{S\tau}(c)$. Otherwise there is some $v \in A_{B\tau}$ such that $t_\tau(v,c) > \bar{v}_\tau q_\tau(v,c) \geq \tilde{v}_\tau(v)\, q_\tau(v,c)$, contradicting (4).

Combining these with the monotonicity of $\tilde{c}_\tau(c)$, we obtain

$$(\bar{v}_\tau - \underline{c}_\tau)\, q_{S\tau}(c) \geq \left[\bar{v}_\tau - \tilde{c}_\tau(c)\right] q_{S\tau}(c) \geq t_{S\tau}(c) - \tilde{c}_\tau(c) q_{S\tau}(c) \geq \alpha_S \frac{\underline{v}_\tau - \underline{c}_\tau}{2},$$

and therefore

$$q_{S\tau}(c) \geq \frac{\alpha_S}{2}\frac{\underline{v}_\tau - \underline{c}_\tau}{\bar{v}_\tau - \underline{c}_\tau} \geq \frac{\alpha_S \kappa_B}{2(r+\kappa_B)},$$

where the last inequality follows from applying the bound from step 1(a).

Then from (11) in Lemma 1,

$$\tilde{c}'_\tau(c) = \frac{1-R_\tau}{1-R_\tau + R_\tau \ell_S q_{S\tau}(c)} \leq \frac{r\tau}{\ell_S q_{S\tau}(c)} \leq \frac{r\tau}{\ell_S \alpha_S \kappa_B / 2(r+\kappa_B)} = \frac{\tau 2r(r+\kappa_B)}{\ell_S \alpha_S \kappa_B}.$$

Now we can see that

$$\frac{y}{2} = \int_{\tilde{c}_\tau(c)\in[\underline{c}_\tau, \underline{c}_\tau + \frac{y}{2}]} \tilde{c}'_\tau(c)\, dc \leq \frac{\tau 2r(r+\kappa_B)}{\ell_S \alpha_S \kappa_B},$$

which is the same as (a).

Step 3: Let $\kappa = \min\{\kappa_B, \kappa_S\}$. We claim that

$$\bar{v}_\tau - \underline{c}_\tau \leq \tau \min\left\{\frac{\kappa_B}{\alpha_B \ell_B}, \frac{\kappa_S}{\alpha_S \ell_S}\right\} \cdot \left(1 + \frac{2r}{\kappa}\right)^3.$$

To prove it, first notice that from step 2(a) and (20), we have

$$\underline{v}_\tau - \underline{c}_\tau = \min\{\underline{v}_\tau, \bar{c}_\tau\} - \underline{c}_\tau + \max\{\underline{v}_\tau - \bar{c}_\tau, 0\} \leq \frac{\tau 4r(r+\kappa_B)}{\ell_S \alpha_S \kappa_B} + \frac{\tau \kappa_S}{\ell_S \alpha_S}.$$

Then from step 1(a),

$$\begin{aligned}
\bar{v}_\tau - \underline{c}_\tau &\leq \frac{r+\kappa_B}{\kappa_B}(\underline{v}_\tau - \underline{c}_\tau) \leq \tau \frac{r+\kappa_B}{\ell_S \alpha_S \kappa_B}\left[\frac{4r(r+\kappa_B)}{\kappa_B} + \kappa_S\right] \\
&= \tau \frac{\kappa_S}{\ell_S \alpha_S}\left(1 + \frac{r}{\kappa_B}\right)\left[1 + \frac{4r}{\kappa_S}\left(1 + \frac{r}{\kappa_B}\right)\right] \\
&\leq \tau \frac{\kappa_S}{\ell_S \alpha_S}\left(1 + \frac{r}{\kappa}\right)\left[1 + \frac{4r}{\kappa}\left(1 + \frac{r}{\kappa}\right)\right] = \tau \frac{\kappa_S}{\ell_S \alpha_S}\left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2 \\
&\leq \tau \frac{\kappa_S}{\alpha_S \ell_S}\left(1 + \frac{2r}{\kappa}\right)^3.
\end{aligned}$$

Similarly, from step 2(b) and step 1(b),

$$\bar{v}_\tau - \underline{c}_\tau \leq \tau \frac{\kappa_B}{\ell_B \alpha_B}\left(1 + \frac{2r}{\kappa}\right)^3.$$

Step 4: Notice that $\kappa_S / (\ell_S(\zeta)\alpha_S)$ is nonincreasing and $\kappa_B / (\ell_B(\zeta)\alpha_B)$ is nondecreasing in ζ, and that they are equal if and only if $\zeta = \zeta_0$. If $\zeta = \zeta_0$, both of them are equal to $K(\zeta_0)/(\alpha_B + \alpha_S)$. Thus step 3 implies the upper bound in the Theorem.

Step 5: Finally, we show that $\bar{v}_\tau - \underline{c}_\tau \geq \tau\kappa$. Consider once again (13) and (14). Since $\ell_B q_{B\tau}$ and $\ell_S q_{S\tau}$ are at most 1, and the ex post IR condition (4) implies $t_{B\tau}(\underline{v}_\tau) \geq \underline{c}_\tau q_{B\tau}(\underline{v}_\tau)$ and $t_{S\tau}(\bar{c}_\tau) \leq \bar{v}_\tau q_{S\tau}(\bar{c}_\tau)$, we have

$$\bar{v}_\tau - \underline{c}_\tau \geq \underline{v}_\tau - \underline{c}_\tau \geq \tau\kappa_B \quad \text{and} \quad \bar{v}_\tau - \underline{c}_\tau \geq \bar{v}_\tau - \bar{c}_\tau \geq \tau\kappa_S.$$

Consequently,

$$\bar{v}_\tau - \underline{c}_\tau \geq \tau \max\{\kappa_B, \kappa_S\} \geq \tau\kappa.$$

Necessity proof:

Take any sequence of ex post IR and IC DBM market equilibria, with $\bar{v}_\tau - \underline{c}_\tau \to 0$ at a linear rate. Then (see footnote 13)

$$\limsup_{\tau\to 0} \frac{\bar{v}_\tau - \underline{c}_\tau}{\tau} < \infty. \tag{33}$$

Step 1: There exist $\alpha_B, \alpha_S, \bar{\tau} > 0$ such that for any $\tau \in (0, \bar{\tau})$,

$$q_{B\tau}(\underline{v}_\tau) \geq \alpha_B \quad \text{and} \quad q_{S\tau}(\bar{c}_\tau) \geq \alpha_S.$$

To prove that, first recall (32) in the Sufficiency proof step 1 (which only requires IC and ex post IR conditions). We have, for any $\tau > 0$,

$$q_{B\tau}(\underline{v}_\tau) \geq \ell_B(\zeta_\tau)\, q_{B\tau}(\underline{v}_\tau) \geq \kappa_B \cdot \frac{\tau}{\underline{v}_\tau - \underline{c}_\tau}.$$

Now notice that (33) implies

$$\limsup_{\tau\to 0} \frac{\underline{v}_\tau - \underline{c}_\tau}{\tau} \leq \limsup_{\tau\to 0} \frac{\bar{v}_\tau - \underline{c}_\tau}{\tau} < \infty$$

so that

$$\liminf_{\tau\to 0} q_{B\tau}(\underline{v}_\tau) \geq \kappa_B \cdot \liminf_{\tau\to 0} \frac{\tau}{\underline{v}_\tau - \underline{c}_\tau} > 0.$$

Therefore, there exist $\alpha_B > 0$ and $\bar{\tau} > 0$ such that $\tau \in (0, \bar{\tau})$ implies $q_{B\tau}(\underline{v}_\tau) \geq \alpha_B$. Similarly one can prove the claim for $q_{S\tau}(\bar{c}_\tau)$.

Step 2: There exist $\alpha_B, \alpha_S, \bar{\tau} > 0$ such that for any $\tau \in (0, \bar{\tau})$,

$$U_{B\tau}(\underline{v}_\tau) \geq \alpha_B \cdot (\underline{v}_\tau - \bar{c}_\tau) \quad \text{and} \quad U_{S\tau}(\bar{c}_\tau) \geq \alpha_S \cdot (\underline{v}_\tau - \bar{c}_\tau).$$

To prove that, first notice that (13) implies, for any $\tau > 0$,

$$U_{B\tau}(\underline{v}_\tau) \geq \ell_B(\zeta_\tau)\, U_{B\tau}(\underline{v}_\tau) = \tau\kappa_B$$

so that

$$\frac{\max\{\underline{v}_\tau - \bar{c}_\tau, 0\}}{U_{B\tau}(\underline{v}_\tau)} \leq \frac{\max\{\underline{v}_\tau - \bar{c}_\tau, 0\}}{\tau\kappa_B} \leq \frac{1}{\kappa_B} \cdot \frac{\bar{v}_\tau - \underline{c}_\tau}{\tau}.$$

Now (33) implies

$$\limsup_{\tau\to 0} \frac{\underline{v}_\tau - \bar{c}_\tau}{U_{B\tau}(\underline{v}_\tau)} < \infty.$$

Therefore, there exist $\alpha_B > 0$ and $\bar{\tau} > 0$ such that $\tau \in (0, \bar{\tau})$ implies $U_{B\tau}(\underline{v}_\tau) \geq \alpha_B \cdot (\underline{v}_\tau - \bar{c}_\tau)$. Similarly one can prove the claim for $U_{S\tau}(\bar{c}_\tau)$.

Step 3: There exist $\alpha_B, \alpha_S, \bar{\tau} > 0$ such that for any $\tau \in (0, \bar{\tau})$, any $v \in [\underline{v}_\tau, 1]$, and any $c \in [0, \bar{c}_\tau]$,

$$U_{B\tau}(v) \geq \alpha_B \cdot (\tilde{v}_\tau(v) - \bar{c}_\tau) \quad \text{and} \quad U_{S\tau}(c) \geq \alpha_S \cdot (\underline{v}_\tau - \tilde{c}_\tau(c)).$$

To prove that, we pick $\alpha_B, \alpha_S, \bar{\tau} > 0$ such that the claims in step 1 and step 2 hold.

By Lemma 1, the Envelope Theorem applies to the IC conditions (2). Thus, for all $\tau \in (0, \bar{\tau})$ and almost all $v \in [\underline{v}_\tau, 1]$, we have $U'_{B\tau}(v) = \tilde{v}'_\tau(v)\, q_{B\tau}(v)$. From the monotonicity of $q_{B\tau}$ and step 1,

$$U'_{B\tau}(v) \geq \tilde{v}'_\tau(v)\, q_{B\tau}(\underline{v}_\tau) \geq \alpha_B \tilde{v}'_\tau(v)$$

for almost all $v \in [\underline{v}_\tau, 1]$. Together with step 2, this implies

$$\begin{aligned} U_{B\tau}(v) &= U_{B\tau}(\underline{v}_\tau) + \int_{\underline{v}_\tau}^{v} U'_{B\tau}(x)\, dx \\ &\geq \alpha_B (\underline{v}_\tau - \bar{c}_\tau) + \alpha_B \int_{\underline{v}_\tau}^{v} \tilde{v}'_\tau(x)\, dx \\ &= \alpha_B \cdot (\tilde{v}_\tau(v) - \bar{c}_\tau). \end{aligned}$$

Similarly one can prove the claim for $U_{S\tau}(c)$. Q.E.D.

Proof of Corollary 1

We only prove the result for the buyers; the proof for the sellers is parallel. Observe that $R_\tau W_{B\tau}(v) = v - \tilde{v}_\tau(v)$. Consequently,

$$\begin{aligned} W^*_B(v) - R_\tau W^+_{B\tau}(v) &= \max\{v - p^*, 0\} - \max\{R_\tau W_{B\tau}(v), 0\} \\ &= \begin{cases} \tilde{v}_\tau(v) - p^* & \text{if} \quad v \geq \underline{v}_\tau \text{ and } v \geq p^* \\ \tilde{v}_\tau(v) - v & \text{if} \quad v \geq \underline{v}_\tau \text{ and } v < p^* \\ v - p^* & \text{if} \quad v < \underline{v}_\tau \text{ and } v \geq p^* \\ 0 & \text{if} \quad v < \underline{v}_\tau \text{ and } v < p^* \end{cases}. \end{aligned}$$

In any of the four cases, we must have $\left| W^*_B(v) - R_\tau W^+_{B\tau}(v) \right| \leq \bar{v}_\tau - \underline{c}_\tau$. It is obvious for the fourth case. For the other three cases, recall that (i) $p^* \in [\underline{c}_\tau, \bar{v}_\tau]$, (ii) $\tilde{v}_\tau(v) \in [\underline{c}_\tau, \bar{v}_\tau]$ under the conditions of the first or second case, and (iii) $v \in [\underline{c}_\tau, \bar{v}_\tau]$ under the conditions of the second or third case.

Moreover,

$$\left| W^+_{B\tau}(v) - R_\tau W^+_{B\tau}(v) \right| = (1 - R_\tau)\, W^+_{B\tau}(v) \leq r\tau.$$

Therefore,

$$\left| W^*_B(v) - W^+_{B\tau}(v) \right| \leq \bar{v}_\tau - \underline{c}_\tau + r\tau.$$

This corollary then follows from Theorem 1. Q.E.D.

Proof of Theorem 2

Recall (23). Since the traders in equilibrium enter voluntarily, the numerators in (23) are non-negative for $v \in A_B$ and $c \in A_S$. Consequently, for active types $W_{B\tau}(v)$ and $W_{S\tau}(c)$ can be bounded from above as

$$W_{B\tau}(v) \leq \frac{\ell_B [q_{B\tau}(v)\, v - t_{B\tau}(v)] - \tau\kappa_B}{\ell_B q_{B\tau}(v)} \leq v - \frac{t_{B\tau}(v)}{q_{B\tau}(v)} - \frac{\tau\kappa_B}{\ell_B},$$

$$W_{S\tau}(c) \leq \frac{t_{S\tau}(c)}{q_{S\tau}(c)} - c - \frac{\tau\kappa_S}{\ell_S}.$$

Now the total welfare of the entering cohort is bounded from above by

$$\begin{aligned}
& b\int_{A_{B\tau}} v dF(v) - s\int_{A_{S\tau}} c dG(c) \\
& - \left[b\int_{A_{B\tau}} \frac{t_{B\tau}(v)}{q_{B\tau}(v)} dF(v) - s\int_{A_{S\tau}} \frac{t_{S\tau}(c)}{q_{S\tau}(c)} dG(c) \right] \\
& - \left(\frac{\tau\kappa_B}{\ell_B} + \frac{\tau\kappa_S}{\ell_S} \right) b\int_{A_{B\tau}} dF(v).
\end{aligned}$$

Obviously, the first term does not exceed the Walrasian welfare rate W_0^*. The second term is zero because in a steady state, $bdF(v)/q_B(v) = B\ell_B d\Phi(v)$, $sdG(c)/q_S(c) = S\ell_S d\Gamma(c)$ and the transfers are balanced, $B\ell_B \int_{v\in A_B} t_B(v)\, d\Phi(v) = S\ell_S \int_{c\in A_S} t_S(c)\, d\Gamma(c)$. Consequently,

$$W_\tau^0 \leq W^{0*} - \left(\frac{\tau\kappa_B}{\ell_B(\zeta)} + \frac{\tau\kappa_S}{\ell_S(\zeta)} \right) b\int_{A_{B\tau}} dF(v).$$

The proof is completed by noting that $\tau\kappa_B/\ell_B(\zeta) + \tau\kappa_S/\ell_S(\zeta) \geq \tau \cdot \min_{\zeta>0} K(\zeta)$. Q.E.D.

Proof of Proposition 1

The proof follows the graphical argument shown in Figure 2. Given τ, the right panel shows the marginal types $\underline{v}$ and $\bar{c}$ in a steady-state equilibrium. The left panel shows the supportable values of the market tightness $\underline{\zeta}$ and $\bar{\zeta}$ that correspond to the given gap $\underline{v} - \bar{c} < 1$. (In general, there can be one, two or more such values.)

Our assumption $M(0,S) = M(B,0) = 0$ implies $\ell_B(\infty) = \ell_S(0) = 0$. It in turn implies

$$\lim_{\zeta\to 0} K(\zeta) = \lim_{\zeta\to\infty} K(\zeta) = \infty, \tag{34}$$

as depicted in the left panel.

Given that (34) holds, a solution ζ to the equation $\tau K(\zeta) = \underline{v} - \bar{c}$ exists if and only if $\underline{v} - \bar{c} \in [\tau \cdot \min_{\zeta>0} K(\zeta), 1)$. Since $\lim_{\tau\to 0} \tau K(\zeta) = 0$ for any $\zeta > 0$, we also must have $\tau \cdot \min_{\zeta>0} K(\zeta) \to 0$ as $\tau \to 0$. It proves that the set of supportable entry gaps $\underline{v} - \bar{c}$ converges to the interval $(0, 1)$.

Now fix any τ such that $\tau \cdot \min_{\zeta>0} K(\zeta) < 1$. Consider the longest interval $[\zeta_{0\tau}, \zeta_{1\tau}]$ such that $\tau K(\zeta_{0\tau}) = \tau K(\zeta_{1\tau}) = 1$ and $\tau K(\zeta) < 1$ for $\zeta \in (\zeta_0, \zeta_1)$. For any $\zeta \in (\zeta_{0\tau}, \zeta_{1\tau})$, $\underline{v}$ and $\bar{c}$ can be found uniquely from (27) and (26) (see Figure 2). Denote these mappings as $\underline{v}_\tau(\zeta)$ and $\bar{c}_\tau(\zeta)$. The equilibrium price p can also be found uniquely from equation (24) or equation (25):

$$p_\tau(\zeta) \equiv \bar{c}_\tau(\zeta) + \frac{\tau \cdot \kappa_S}{\ell_S(\zeta)} \tag{35}$$

$$(= \underline{v}_\tau(\zeta) - \frac{\tau \cdot \kappa_B}{\ell_B(\zeta)}). \tag{36}$$

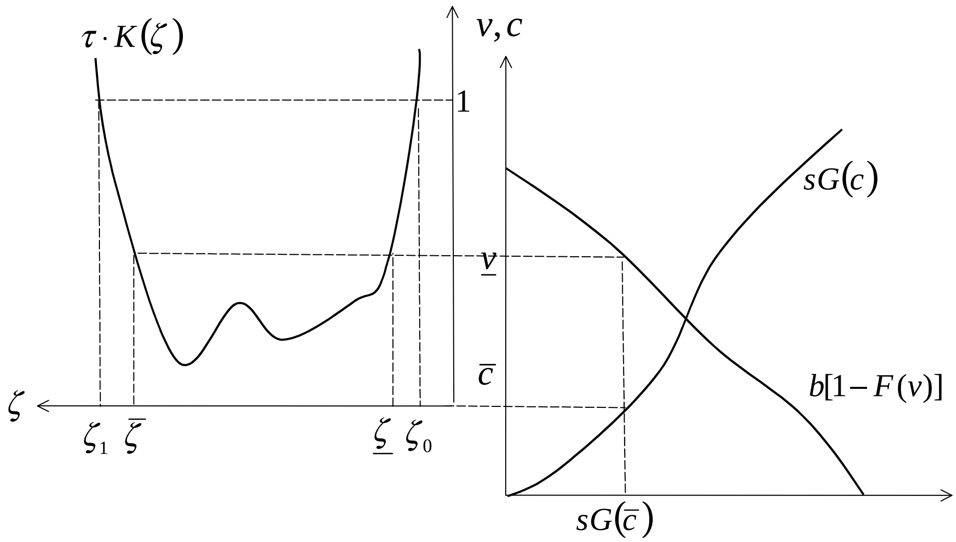

Figure 2: Construction of a full-trade equilibrium of the k-double auction

This formally defines a continuous mapping $p_\tau(\cdot)$ of $[\zeta_{0\tau}, \zeta_{1\tau}]$ into $\mathbb{R}_+$. Consequently, its image is a closed interval that contains the points $p(\zeta_{0\tau})$ and $p(\zeta_{1\tau})$; and the set of supportable equilibrium prices contains this interval. The definitions of $\zeta_{0\tau}$ and $\zeta_{1\tau}$ imply that $\zeta_{0\tau} \to 0$ and $\zeta_{1\tau} \to \infty$ as $\tau \to 0$. Now $\bar{c}_\tau(\zeta_{1\tau}) = 0$ for all τ and $\ell_S(\zeta_{1\tau}) \to \infty$ as $\tau \to 0$, therefore (35) implies that $\lim_{\tau\to 0} p_\tau(\zeta_{1\tau}) = 0$. Similarly, $\underline{v}_\tau(\zeta_{0\tau}) = 1$ for all τ and $\ell_B(\zeta_{0\tau}) \to \infty$ as $\tau \to 0$, so that (36) implies that $\lim_{\tau\to 0} p_\tau(\zeta_{0\tau}) = 1$. This proves that the set of supportable equilibrium price converges to $(0,1)$. Q.E.D.

Proof of Corollary 5

Having Proposition 1, it is sufficient to prove the necessity of the condition $\tau \cdot \kappa_B/\ell_B(\zeta) + \tau \cdot \kappa_S/\ell_S(\zeta) < 1$. The fact that $W_B(v) \geq 0$ and $W_S(c) \geq 0$ for all active types implies that $\forall v \in A_B, c \in A_S$,

$$\ell_B(\zeta)\int [q(v,c)v - t(v,c)]d\Gamma(c) \geq \tau \cdot \kappa_B, \quad \ell_S(\zeta)\int [t(v,c) - q(v,c)c]d\Phi(v) \geq \tau \cdot \kappa_S$$

and hence

$$\frac{\tau \cdot \kappa_B}{\ell_B(\zeta)} + \frac{\tau \cdot \kappa_S}{\ell_S(\zeta)} \leq \int\int (v-c)d\Phi(v)d\Gamma(c) < 1.$$

Q.E.D.

Proof of Theorem 3

We derive a system of equations characterizing the set of two-step equilibria. But before doing so, it is convenient to introduce additional notation. In a two-price equilibrium, the buyers with $v > \hat{v}$ who submit the high bid price $\bar{p}$, trade with any seller they meet. Buyers with $v \in [\underline{v}, \hat{v}]$, who submit the low bid price $\underline{p}$, trade only with sellers having $c < \hat{c}$, who submit $\underline{p}$; their probability of trading is equal to $\Gamma(\hat{c})$. Similarly sellers with $c < \hat{c}$ trade with any buyer they meet, and sellers with $c \in [\hat{c}, \bar{c}]$ trade only with those buyers with $v > \hat{v}$; their probability of trading is equal to $1 - \Phi(\hat{v})$.

In the equilibria to be constructed $\Gamma(\hat{c})$ and $1 - \Phi(\hat{v})$ will converge to 0 as τ goes to 0, so it is convenient to divide them by τ:

$$\pi_B \equiv \frac{1 - \Phi(\hat{v})}{\tau}, \qquad \pi_S \equiv \frac{\Gamma(\hat{c})}{\tau}.$$

Since $\underline{v}$-buyers and $\bar{c}$-sellers are indifferent between entering or not, we have

$$\ell_B \pi_S (\underline{v} - \underline{p}) = \kappa_B, \tag{37}$$

$$\ell_S \pi_B (\bar{p} - \bar{c}) = \kappa_S. \tag{38}$$

Since $\hat{v}$-buyers are indifferent between biding $\underline{p}$ or $\bar{p}$, and $\hat{c}$-sellers are indifferent between asking $\underline{p}$ or $\bar{p}$, we have

$$\tau\pi_S[\tilde{v}(\hat{v}) - \underline{p}] = \tau\pi_S \left\{\tilde{v}(\hat{v}) - [(1-k)\underline{p} + k\bar{p}]\right\} + (1 - \tau\pi_S)\left[\tilde{v}(\hat{v}) - \bar{p}\right], \tag{39}$$

$$\tau\pi_B[\bar{p} - \tilde{c}(\hat{c})] = \tau\pi_B \left\{[(1-k)\underline{p} + k\bar{p}] - \tilde{c}(\hat{c})\right\} + (1 - \tau\pi_B)\left[\underline{p} - \tilde{c}(\hat{c})\right]. \tag{40}$$

Since the utility equations (8), (9) still hold here, we have

$$\hat{W}_B = (\hat{v} - \underline{v}) \frac{m(\zeta)\pi_S}{\zeta\,(1 - R_\tau) + R_\tau m(\zeta)\pi_S} \tag{41}$$

$$\hat{W}_S = (\bar{c} - \hat{c}) \frac{m(\zeta)\pi_B}{1 - R_\tau + R_\tau m(\zeta)\pi_B}. \tag{42}$$

where we denoted $\hat{W}_B \equiv W_B(\hat{v})$ and $\hat{W}_S \equiv W_S(\hat{c})$.

To complete the description of a two-step equilibrium, the indifference conditions are supplemented with steady-state mass balance conditions for each interval of types. Here, it suffices to require that the total inflows into the intervals $[\underline{v}, 1]$ and $[0, \bar{c}]$ are balanced with the outflows,

$$b\left[1 - F(\underline{v})\right] = Sm(\zeta)\left[\pi_S + \pi_B(1 - \tau\pi_S)\right], \tag{43}$$

$$sG(\bar{c}) = Sm(\zeta)\left[\pi_B + \pi_S(1 - \tau\pi_B)\right], \tag{44}$$

and that the inflows into the intervals $v \in [\hat{v}, 1]$ and $[0, \hat{c}]$ are also balanced with the outflows,

$$b[1 - F(\hat{v})] = Sm(\zeta)\pi_B, \tag{45}$$

$$sG(\hat{c}) = Sm(\zeta)\pi_S. \tag{46}$$

We also define the price spread,

$$a_0 \equiv \bar{p} - \underline{p}.$$

Then equations (37) through (46) form a 10-equation system with 12 endogenous variables $\{\underline{p}, a_0, \zeta, \underline{v}, \bar{c}, \hat{v}, \hat{c}, \pi_B, \pi_S, S, \hat{W}_B, \hat{W}_S\}$. This system does characterize an equilibrium. Indeed, one can easily see that the buyers with $v \in (\hat{v}, 1]$ strictly prefer to bid $\bar{p}$, the buyers with $v \in (\underline{v}, \hat{v})$ strictly prefer to bid $\underline{p}$, and the buyers with $v \in [0, \underline{v})$ strictly prefer not to enter. Similar remark applies for sellers.

Since we have two degrees of freedom, we can fix some $\zeta > 0$ and $a_0 \in (a, 1)$ and then let equations (37) - (46) determine $\{\underline{p}, \underline{v}, \bar{c}, \hat{v}, \hat{c}, \pi_B, \pi_S, S, \hat{W}_B, \hat{W}_S\}$. We claim that a solution exists for small enough τ and r. To see this, one can check that when $\tau = r = 0$, we have a (unique) solution with $\underline{p}$ implicitly determined by $b[1 - F(\underline{p} + a_0)] = sG(\underline{p})$, and all other variables given by

$$\bar{c} = \underline{p}, \;\; \underline{v} = \bar{p} = \underline{p} + a_0, \;\; \pi_B = \frac{\kappa_S}{m(\zeta)a_0}, \;\; \pi_S = \frac{\kappa_B \zeta}{m(\zeta)a_0}, \;\; S = \frac{sG(\underline{p})a_0}{\kappa_B\zeta + \kappa_S},$$

$$1 - F(\hat{v}) = \frac{[1 - F(\bar{p})]\kappa_S}{\kappa_B\zeta + \kappa_S}, \;\; G(\hat{c}) = \frac{G(\underline{p})\kappa_B\zeta}{\kappa_B\zeta + \kappa_S}, \;\; \hat{W}_B = \hat{v} - \bar{p}, \;\; \hat{W}_S = \underline{p} - \hat{c}.$$

One can also check that the Jacobian evaluated at $\tau = r = 0$ is not zero.[18] Therefore the Implicit Function Theorem applies. Because $\bar{p} - \underline{p} \equiv a_0 > a$, there exists a two-step equilibrium with $\bar{p} - \underline{p} > a$ when τ and r are small enough. Moreover, since $\underline{v} \to \bar{p}$ and $\bar{c} \to \underline{p}$ as $(\tau, r) \to (0, 0)$, the spread $\underline{v} - \bar{c}$ is also bounded below by a. It follows that the associated total ex ante surplus W^0 is bounded away from the Walrasian total ex ante surplus W^{0*}. Q.E.D.

[18]The Mathematica® notebook that contains the evaluation of the Jacobian is available at artyom239.googlepages.com.