The Choice of the Number of Varieties: Justifying Simple Mechanisms

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Abstract

We study a mechanism designer’s trade-off between the complexity level and optimality level of a mechanism. While our methodology and techniques apply to a much larger class of mechanism design problems, we restrict our presentation within the framework of Mussa and Rosen (1978) quality differentiation, in which a monopolist offers a finite number \( n \) of varieties, instead of the fully optimal spectrum. We prove that the marginal benefit of adding one more variety monotonically vanishes at the cubic rate \( O(n^{-3}) \). This extremely fast rate suggests that lengthening the menu, or more generally, complicating a mechanism, is usually not worthwhile.

Keywords: Quality differentiation, Monopoly pricing, Simple mechanisms

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1 Introduction

Mechanism design theory has now become a classic and far-reaching branch in economics. It has been used to derive, for example, optimal income taxation scheme (Mirrlees (1971)),

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optimal nonlinear pricing scheme (Maskin and Riley (1984)), optimal quality differentiation (Mussa and Rosen (1978)), among many others. While these theoretical solutions of optimal mechanisms have been well known, in reality however people embrace much simpler mechanisms, like, an income taxation scheme with several tax bands and several marginal tax rates, a multipart tariff with a small number of "parts", and a quality-price scheme with only a few quality-differentiated varieties.

This paper studies how well a simple mechanism can approximate the fully optimal mechanism, and how the "complexity level" of a mechanism can be traded off with the "optimality level" of it. To make our presentation concrete, we will restrict attention to a particular application: monopolistic quality differentiation problem. In this application, there is a natural measure for the complexity level of a mechanism, namely, the number of quality-differentiated varieties to be offered. It should however be noted that the methodology and techniques developed in this paper are general enough to be applied to the much larger class of mechanism design (or principal-agent) problems considered in Fudenberg and Tirole (1991) Chapter 7, or in Guesnerie and Laffont (1984).1 Our analysis suggests that simple mechanisms are, in a sense, justified against the theoretical optimal but complicated mechanisms.

More precisely, we will consider the framework of Mussa and Rosen (1978) quality differentiation. In this framework, a monopolist is uninformed about its customers' preferences over quality (or types), but it can produce and offer a spectrum of quality-differentiated varieties to separate different types of its customers. The optimal spectrum involves a continuum of quality-differentiated varieties, tailor-made for each consumer type. However, if the monopolist desires to offer a finite number $n$ of varieties only, it would design a discrete offering (i.e. a menu of a finite number of quality-price choices), in order to maximize profit subject to the number of varieties $n$. There would then be a "constrained profit" $\Pi_n$ for each $n$. Our main task is to characterize the properties of the profit sequence $\{\Pi_n\}_{n=1}^{\infty}$. We also consider the setting with a fixed cost of developing each variety, which endogenizes the number of varieties.

Our main result is that the marginal benefit of adding an extra variety in the menu (i.e. $\Pi_{n+1} - \Pi_n$) monotonically vanishes at the "cubic rate", i.e. $\Pi_{n+1} - \Pi_n$ is strictly decreasing in $n$, and is of order $1/n^3$. Such an extremely fast convergence rate suggests that the monopolist can largely reduce the complexity of its menu by giving up a tiny amount of optimality. As a corollary, the existence of a tiny marginal cost of developing extra varieties (which is the cost of complexity) can plausibly justify the optimal number of varieties (or optimal complexity level) to be quite small. As another corollary, we also show that the “uncaptured profit”

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1In particular, there is one principal and one agent, and the agent has one-dimensional private information.
(i.e. $\Pi_\infty - \Pi_n$) due to discrete offering vanishes at the "quadratic rate" (i.e. of order $1/n^2$), meaning that discrete offering can well approximate the fully optimal scheme. Numerical examples will also be presented.

Although the questions we ask are of great practical importance, the relevant literature is relatively small. As Neeman (2003) pointed out, "economic theory typically only distinguishes between optimal and sub-optimal outcomes. By and large, there is no attempt to quantify how far from optimality are sub-optimal outcomes, allocations, or institutions."

Despite of it, the literature still has some interesting results about the performance of simple but sub-optimal mechanisms. Wilson (1993) and Miravete (2007) are concerned with nonlinear pricing. Wilson (1993) Section 8.3 presents a rate of convergence result showing the approximate optimality of $n$-part tariffs. Miravete (2007) uses a large sample of independent cellular telephone markets to structurally estimate a monopolistic nonlinear pricing model. His estimates suggests that "firms should only offer few tariff options if the product development costs of designing them are non-negligible."

Wilson (1989) and McAfee (2002) analyze rationing system (of e.g. electricity). McAfee (2002) shows that, if some regularity conditions hold, a social planner can divide the users into only two priority classes (high priority customers and low priority customers) such that an associated two-class priority pricing scheme achieves at least half of the social gains of using infinitely many priority classes. Wilson (1989) shows that the losses due to using only $n$ priority classes are of order $1/n^2$.

In the context of procurement contracting, Rogerson (2003) considers "Fixed Price Cost Reimbursement (FPCR) menus", that is, two-item menus where one item is a cost-reimbursement contract and the other item is a fixed-price contract, of which the principal allows the agent to pick one. He shows that, if the agent’s utility is quadratic and the agent’s type is distributed uniformly, then "the optimal FPCR menu always captures at least three-quarters of the gain that the optimal complex menu achieves". Chu and Sappington (2007) relax the assumption of uniform distribution, and show that a menu of two options, namely, a cost-reimbursement contract and a linear cost sharing contract, can always secure at least 73 percent of the gain.

In the context of selling an indivisible object, Neeman (2003) shows the effectiveness of a simple selling mechanism, namely, English auction, for general private-values environments, and for private-values environments where bidders’ valuations are non-negatively correlated.

In the context of multiproduct pricing, Chu, Leslie, and Sorensen (2009) show that a simple bundle-size pricing are often nearly optimal. With surprisingly few prices a multiproduct firm can obtain 99% of the profit that would be earned by optimal mixed bundling.

The rest of the paper is organized as follows. Section 2 sets up the environment and

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2 This result is analogous to our Corollary 6 (quadratic rate result). See our discussion in the text.
provides the standard solution in the literature. Section 3 characterizes the optimal discrete offering and provides a preliminary analysis of the constrained profit conditional on the number of varieties. Section 4 presents the properties of the constrained profit sequence, especially the diminishing marginal benefit and the rate of convergence results. Section 5 studies a solvable class of examples, revealing that the convergence rates we provide are tight. Section 6 concludes. The proofs we do not provide in the text are in Appendix.

2 The Model

2.1 Environment

Consider the Mussa and Rosen (1978) monopolistic quality differentiation environment. A commodity can be produced by a monopolist in a spectrum of varieties. The hedonic attributes of the varieties are characterized by a one-dimensional non-negative quality index $q$. The consumers have unit demand for the commodity, i.e. every consumer chooses to buy either 0 or 1 unit. A consumer who decides to buy the commodity must choose one offered variety to pick up. The higher the quality index of a variety, the higher the willingness-to-pay of a consumer is.

Consumers are heterogeneous in their types, which are indexed by $t$. The utility of a type $t$ consumer who buys a variety with quality $q$ and pays the price $p$ is represented as $tq - p$. If the consumer does not buy a unit, her utility is zero. The distribution of $t$ is characterized by a cumulative distribution function $F(\cdot)$, whose support is a compact interval $[\underline{t}, \bar{t}]$. We assume that $F(\cdot)$ admits a density function $f(\cdot)$ which is strictly positive and continuously differentiable on $(\underline{t}, \bar{t})$. The type of a consumer is the consumer’s private information. The monopolist only knows the prior distribution $F(\cdot)$.

The unit cost of producing variety with quality $q$ is denoted as $c(q)$. The function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is twice differentiable and has the following properties:

$$c(0) = 0, \quad c'(0) < \bar{t} < c'(\infty),$$

for all $q > 0$, $c'(q) > 0$, $c''(q) > 0$. (1)

2.2 Unconstrained solution

If the monopolist can costlessly establish as many varieties as it wants, the problem it faces is exactly the standard problem studied by Mussa and Rosen (1978). That is, the monopolist solves

$$\max_{q(\cdot) \geq 0 \mu(\cdot)} \int_{\underline{t}}^{\bar{t}} [p(t) - c(q(t))] dF(t)$$

(2)
subject to
\[ t q(t) - p(t) \geq 0 \quad \forall t \in [\underline{t}, \bar{t}] \]  
(3)
\[ t q(t) - p(t) \geq t q(t') - p(t') \quad \forall t, t' \in [\underline{t}, \bar{t}]. \]  
(4)

The functions \( p(t) \) and \( q(t) \) specify the monopolist’s choice of price and quality for consumers with type \( t \). The objective function in (2) is the (per customer) profit. The constraint (3) is the individual rationality (IR) constraint, which exists because every consumer always has the choice to buy nothing, pay nothing and get the reservation utility zero. The constraint (4) is the incentive compatibility (IC) constraint, which arises from the fact that the consumers’ types are private information.

Since the problem above has no constraint on the number of varieties to be offered (as opposed to the “constrained problem” or “constrained program” later), and hence the standard solution in general involves a continuum of varieties, let us call the maximized value of the problem (2) the “unconstrained profit”, which is denoted as \( \Pi_\infty \).

Adopting the standard technique of solving this kind of problem\(^3\), the monopoly unconstrained program is reduced to
\[
\max_{q(\cdot) \geq 0} \int_{\underline{t}}^{\bar{t}} [J(t)q(t) - c(q(t))] \, dF(t)
\]  
(5)
subject to
\[ q(t) \text{ is nondecreasing in } t, \]

where \( J(t) \equiv t - \frac{1 - F(t)}{f(t)} \) is the “virtual type function”. Assume that \( J(\cdot) \) is strictly increasing, which is a standard regularity condition in the literature. Then the monotonicity constraint of \( q(\cdot) \) is not binding, and the optimal solution of \( q(\cdot) \) is such that \( J(t) = c'(q(t)) \) whenever type \( t \) is allocated a unit of the commodity. For notational convenience later, let us assume \( c'(0) > J(\underline{t}) \), which means that some very low types of consumers will not be allocated a unit of the commodity. Then the unconstrained profit \( \Pi_\infty \) is written as:

**Proposition 1 (Mussa and Rosen, 1978)**

\[ \Pi_\infty = \int_{\underline{t}}^{\bar{t}} [J(t)q_\infty(t) - c(q_\infty(t))] \, dF(t) \]

\(^3\)See for example Fudenberg and Tirole (1991), Chapter 7.
where the optimal continuous offering $q_\infty(\cdot)$ is defined by

$$q_\infty(t) = \begin{cases} 
0 & \text{if } t \leq t_* \\
c'^{-1}(J(t)) & \text{if } t_* \leq t \leq \bar{t}
\end{cases}$$

and the lowest allocated type $t_*$ is defined by $c'(0) = J(t_*)$.

### 3 The Constrained Program

The solution of the unconstrained program in the previous section involves uncountably many varieties to be offered. In the reality, it is hard to imagine that a monopolist would offer such a complicated menu of varieties. In fact, a tiny but positive fixed cost of developing an extra variety would certainly lead the monopolist not to offer infinite varieties, because it would involve an infinite amount of development costs.

It is interesting to investigate whether a small cost of developing varieties leads the monopolist to choose to offer a small number of varieties. Putting this question in another way, we would ask whether offering just a few varieties would allow the monopolist to yield a large fraction of $\Pi_\infty$, and whether offering one extra variety starting from a few varieties raises the constrained profit only stingily. It brings us to study the properties of the constrained profit $\Pi_n$ given some number of varieties $n$.

More concretely, if there is a (per customer) fixed cost $k$ of developing each variety, the number of varieties $n$ is an extra choice variable in the monopoly problem. We can decompose the whole monopoly problem into two stages. In the first stage, given any arbitrary number $n$ of varieties, the monopoly chooses the optimal offering and gets the constrained (gross) profit $\Pi_n$. In the second stage, the monopoly chooses the optimal $n^*$ to maximize the net profit ($\Pi_n - nk$). Hence, we can understand the relationship between $n^*$ and $k$ through studying the properties of the sequence $\Pi_n$.\footnote{The cost of developing $n$ varieties being linear in $n$ is non-crucial. It could also take the form $K(n)$. However, if the cost of developing a variety depends on the quality of that variety, the decomposition of the monopoly problem is invalid. Nevertheless, even in that case the properties of the sequence $\{\Pi_n\}_{n=1}^{\infty}$ are of great interest because it reveals the approximate optimality of simple discrete offering.} We call the aforementioned first stage the “constrained program”.

For the constrained program given $n$, there can be at most $n$ quality-differentiated varieties. Thus the $q(t)$ function can take at most $n$ values except zero. The IR, IC and non-negativity constraints still remain. Therefore the constrained program can be written as

$$\Pi_n = \max_{q(t) \geq 0} \int_0^\bar{t} [J(t)q(t) - c(q(t))] \, dF(t)$$  \hspace{1cm} (6)

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$$\Pi_n = \max_{q(t) \geq 0} \int_0^\bar{t} [J(t)q(t) - c(q(t))] \, dF(t)$$  \hspace{1cm} (6)
subject to

\[ q(t) \text{ is nondecreasing in } t, \]
\[ q(t) \text{ takes at most } n \text{ values except zero.} \]  

(7)

It is nothing but the unconstrained program adding the constraint (7).

### 3.1 The optimal discrete offering

Now we are ready to analyze the optimal discrete offering in a constrained monopoly problem, given the number of varieties \( n \). First, it is not hard to show, under our assumptions, that the constrained program (6) has a solution, and the monotonicity constraint of \( q(\cdot) \) is not binding. Moreover, from the monotonicity of \( q(\cdot) \) and the constraint (7), \( q(\cdot) \) must be a nondecreasing \( n \)-step function. Hence we can rewrite the constrained program (6) as a finite dimensional problem:

\[
\max_{q_1, \cdots, q_n \in \mathbb{R}_+; t_1, \cdots, t_n \in [\underline{t}, \bar{t}]} \left\{ \int_{t_1}^{t_2} [J(t)q_1 - c(q_1)] dF(t) + \cdots + \int_{t_{n-1}}^{t_n} [J(t)q_{n-1} - c(q_{n-1})] dF(t) + \int_{t_n}^{\bar{t}} [J(t)q_n - c(q_n)] dF(t) \right\}. 
\]

(8)

The first-order necessary conditions of (8) with respect to \((q_1, \cdots, q_n; t_1, \cdots, t_n)\) can be written as two first-order difference equations

\[
c'(q_i) = \frac{\int_{t_i}^{t_{i+1}} J(t) dF(t)}{F(t_{i+1}) - F(t_i)} \quad \forall i = 1, \cdots, n, \quad \text{(9)}
\]

\[
J(t_i) = \frac{\int_{q_{i-1}}^{q_i} c'(q) dq}{q_i - q_{i-1}} \quad \forall i = 1, \cdots, n, \quad \text{(10)}
\]

and two boundary conditions

\[
q_0 = 0, \quad \text{(11)}
\]

\[
t_{n+1} = \bar{t}. \quad \text{(12)}
\]

Therefore, the optimal discrete offering \((q_1, \cdots, q_n; t_1, \cdots, t_n)\) in a constrained program can be characterized as the solution of a two-equation system of difference equations. This system does not have closed-form solution except special cases (see Section 5). It is not surprising that both the difference equations (9) and (10) converge to the unconstrained optimal offering formula \( J(t) = c'(q(t)) \) as the consecutive \( t_i \)'s and \( q_i \)'s become closer and closer.
Proposition 2 There exists a solution \((q_1, \cdots, q_n; t_1, \cdots, t_n)\) to the difference equations (9) and (10), coupled with the boundary conditions (11) and (12), such that

\[
\Pi_n = \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} [J(t)q_i - c(q_i)] dF(t).
\]

Proof. First, in problem (6), the monotonicity constraint of \(q(\cdot)\) is not binding. To see this, suppose that \(q(\cdot)\) takes values \(q_1, \cdots, q_n\) except zero, with \(0 \leq q_1 \leq \cdots \leq q_n\). Let the monotonicity constraint of \(q(\cdot)\) be ignored. Then, to maximize the objective function in (6), \(q(\cdot)\) must satisfy: for every \(t \in [t_i, \bar{t}_i]\), \(q(t)\) is some \(q_i\) that maximize \(J(t)q_i - c(q_i)\) among \(q_1, \cdots, q_n\). Now since the function \(H(q,t) \equiv J(t)q - c(q)\) satisfies increasing differences in \((q,t)\), this \(q(t)\) must be nondecreasing in \(t\).

Second, the equivalent version of the constrained program (8) has a solution, and hence \(\Pi_n\) is well-defined. To see this, notice that the objective function in (8) is continuous in \((q_1, \cdots, q_n; t_1, \cdots, t_n)\). Moreover, \(q_n\) can be without loss restricted to be below some large upper bound, because our assumption (1) implies that \(\bar{t}q < c(q)\) for large enough \(q\). Then the constraint set is compact, and from Weierstrass Theorem a maximizer exists.

Third, our previous analysis reveals that any solution to (8) satisfies (9), (10), (11) and (12). \(\blacksquare\)

The pattern of an optimal discrete offering is sketched in Figure 1: the optimal discrete offering of a constrained program is a step-function approximation of the optimal continuous offering of the unconstrained program.\(^5\)

Once the solution of \((q_1, \cdots, q_n; t_1, \cdots, t_n)\) is characterized, the solution of \((p_1, \cdots, p_n)\) directly follows. Namely,

\[
p_1 = t_1q_1
\]

and

\[
p_i = p_{i-1} + t_i(q_i - q_{i-1}) \quad \forall i = 2, \cdots, n.
\]

The first one is from the fact that a type \(t_1\) consumer will be indifferent between buying the variety with quality \(q_1\) or buying nothing. The second one is due to the fact that a type \(t_i\) consumer will be indifferent between buying the variety with quality \(q_i\) or with quality \(q_{i-1}\).

\(^5\)Even if the virtual type function \(J\) is not monotone so that the monotonicity of \(q(\cdot)\) might be binding, (9) – (12) are still necessary conditions for the constrained program. Hence our analysis is still valid.
3.2 Visualizations of constrained and unconstrained profit

Recall that \( q_\infty(\cdot) \), as defined in Proposition 1, is the optimal continuous offering of the unconstrained monopoly problem. And we will use \((q_1,n,q_2,n,\ldots,q_n,n; t_1,n,t_2,n,\ldots,t_n,n)\) to denote the optimal discrete offering of the constrained monopoly problem given the number of varieties \( n \). Let us also concisely use a step function \( q_n(t) \) to describe the optimal discrete offering, in particular,

\[
q_n(t) = \begin{cases} 
q_{0,n} & \text{if } t \in [t, t_{1,n}) \\
q_{1,n} & \text{if } t \in [t_{1,n}, t_{2,n}) \\
\vdots & \vdots & \vdots \\
q_{n,n} & \text{if } t \in [t_{n,n}, t_{n+1,n}] \equiv [t_{n,n}, \bar{t}].
\end{cases}
\]

For notational simplicity, we will write \( q_1, q_2, \ldots, q_n \) and \( t_1, t_2, \ldots, t_n \) instead of \( q_1,n,q_2,n,\ldots,q_n,n \) and \( t_1,n,t_2,n,\ldots,t_n,n \) when no confusion would be raised.

For convenience, we define

\[
H(q,t) \equiv J(t)q - c(q).
\] (13)

Then (10) can be written as

\[
H(q_i,t_i) = H(q_{i-1},t_i) \quad \forall i = 1, \ldots, n,
\] (14)
and the unconstrained profit $\Pi_\infty$ and the constrained profit $\Pi_n$ can be written as

$$\Pi_\infty = \int_t^\bar{t} H(q_\infty(t),t)dF(t)$$  \hspace{1cm} (15)$$

and

$$\Pi_n = \int_t^\bar{t} H(q_n(t),t)dF(t)$$
$$= \int_{t_1}^{t_2} H(q_1,t)dF(t) + \int_{t_2}^{t_3} H(q_2,t)dF(t) + \cdots + \int_{t_n}^{\bar{t}} H(q_n,t)dF(t).$$  \hspace{1cm} (16)$$

Fixing any $q_i$, the slope of $H(q_i,t)$ with respect to $t$ is

$$\frac{\partial H(q_i,t)}{\partial t} = J'(t)q_i$$

which is positive. Besides, the slope of $H(q_\infty(t),t)$ with respect to $t$ is

$$\frac{dH(q_\infty(t),t)}{dt} = J'(t)q_\infty(t) + J(t)q'_\infty(t) - c'(q_\infty(t))q'_\infty(t)$$
$$= J'(t)q_\infty(t)$$

which is positive unless $t \leq t_*$ (which implies $q_\infty(t) = 0$).

Now, we can nicely visualize $\Pi_\infty$ and $\Pi_n$ in a single diagram.\footnote{Analyzing the social planner’s problems or the perfect information monopolist problems (both constrained and unconstrained) only amounts to replacing $J(t)$ by $t$. All our results can be easily adapted there.} First note that for any $i$, the two curves $H(q_i,t)$ and $H(q_{i-1},t)$ plotted against $t$ must cross only once at $t = t_i$; and hence when plotted against $F(t)$, they must cross only once at $F(t) = F(t_i)$. Then note that due to the optimal nature of the function $q_\infty(t)$, we must have

$$H(q_\infty(t),t) = \max_q \{H(q,t)\} \hspace{1cm} \forall t \in [t_*, \bar{t}].$$

In other words, the curve $H(q_\infty(t),t)$ plotted against $t$ or against $F(t)$ is the upper envelope of all the curves $H(q,t)$ with various values of $q$. The ideas are shown by Figure 2, in which $n = 3$. From (15), it is clear that $\Pi_\infty$ is the area below the bold curve $H(q_\infty(t),t)$ in Figure 2. Moreover, (16) says that $\Pi_n$ is represented as the shaded area.

An important insight shed from Figure 2 is that each of the varieties offered helps capturing the unconstrained profit $\Pi_\infty$ in a first-order sense. The uncaptured profit is therefore of second or higher order. This idea is helpful for understanding the rate of
Figure 2: Visualizations of the unconstrained profit $\Pi_\infty$ and the constrained profit $\Pi_n$

convergence results presented in the next section.

4 Properties of the Constrained Profit Sequence $\{\Pi_n\}_{n=1}^\infty$

This section presents the general properties of the sequence of constrained profit $\{\Pi_n\}_{n=1}^\infty$, and their implication on the monopoly choice of the number of varieties $n^*$.

4.1 Basic properties and diminishing marginal benefit

Define $\Pi_0$ as 0. The sequence of constrained profit $\{\Pi_n\}_{n=1}^\infty$ has the following properties.

**Proposition 3** (i) $\Pi_n > \Pi_{n-1}$ for every $n = 1, 2, \ldots$; (ii) $\lim_{n \to \infty} \Pi_n = \Pi_\infty$, where $\Pi_\infty$ is characterized in Proposition 1; and (iii) $\lim_{n \to \infty} (\Pi_n - \Pi_{n-1}) = 0$.

**Proof.** An increase in $n$ (i.e. more varieties allowed) weakens the constraints of the constrained program, thus $\Pi_n \geq \Pi_{n-1}$ for every $n$. Moreover, this inequality must be strict because the optimal discrete offering in a $n$-variety program must involve $n$ different varieties. To see this, suppose $q(\cdot)$ takes only $n-1$ values except zero in a $n$-variety program. Because the optimal continuous offering $q_\infty(\cdot)$ (defined in Proposition 1) is strictly increasing, some $\hat{q}(\cdot)$ function that takes $n$ values except zero can approximate $q_\infty(\cdot)$ better than $q(\cdot)$ does, i.e. $|q_\infty(t) - \hat{q}(t)| \leq |q_\infty(t) - q(t)|$ for all $t \in [\underline{t}, \bar{t}]$ and $|q_\infty(t) - \hat{q}(t)| < |q_\infty(t) - q(t)|$ for all $t$ in some open subset of $[\underline{t}, \bar{t}]$. Since the objective function’s integrand $H(q, t) \equiv J(t) q - c(q)$
is single-peaked in $q$ and $q_\infty(t)$ is the maximizer of $H(\cdot, t)$, we see that the $n$-variety offering $\hat{q}(\cdot)$ yields strictly higher profit than the $(n - 1)$-variety offering $q(\cdot)$ does. It proves (i).

Now the sequence $\{\Pi_n\}_{n=1}^\infty$ is increasing and bounded from above by $\Pi_\infty$. Thus it has a finite limit, and $\lim_{n \to \infty} \Pi_n \leq \Pi_\infty$. To see the equality in (ii), notice that, as an increasing function, $q_\infty(\cdot)$ can be arbitrarily well approximated by nondecreasing finite-step functions.

Finally, since any convergent sequence in Euclidean space is also a Cauchy sequence, $(\Pi_n - \Pi_{n-1})$ converges to zero as $n$ goes to infinity. It proves (iii). $\blacksquare$

In Appendix we also prove that the marginal constrained profit $\Pi_n - \Pi_{n-1}$ is strictly decreasing in $n$. While one might think this is a natural property, it is by no means obvious. (Note that adding one more variety would bring about an optimal adjustment of all previously offered variety.) It is a very nice “diminishing marginal benefit” property, which is interesting on its own. We state it as a separate theorem.

**Theorem 4 (Diminishing marginal benefit)** The increments of $\{\Pi_n\}_{n=1}^\infty$ are decreasing, i.e. $\Pi_{n+1} - \Pi_n < \Pi_n - \Pi_{n-1}$ for every $n = 1, 2, \ldots$.

To understand the idea behind the proof, let us sketch the proof for the inequality $\Pi_3 - \Pi_2 < \Pi_2 - \Pi_1$, or equivalently $\Pi_1 + \Pi_3 < 2\Pi_2$. The shaded areas in the left and middle panels of Figure 3 visualize $\Pi_1$ and $\Pi_3$ respectively. The right panel of Figure 3 visualizes the sum $\Pi_1 + \Pi_3$. In this right panel, $\Pi_1$ is the area below the dashed curve, and $\Pi_3$ is the area below the bold solid curve. A “2” in an area indicates that that area should be counted twice because that area occurs in both $\Pi_1$ and $\Pi_3$. Similarly, a “1” in an area indicates that that area should be counted only once because that area occurs in either $\Pi_1$ or $\Pi_3$, not both. Next we will show that $2\Pi_2$ must be greater than the area indicating $\Pi_1 + \Pi_3$. To do this, we only need to construct two (suboptimal) 2-variety menus such that the sum of those two corresponding suboptimal 2-variety profits is larger than $\Pi_1 + \Pi_3$. The following procedure will do. First, rank all the qualities involved in $\Pi_1$ and $\Pi_3$. For the example shown in Figure 3, this ranking is $q_{1,3} < q_{2,3} < q_{1,1} < q_{3,3}$. Then, collect those with odd ranking in one menu and those with even ranking in another menu. For the current example, the two menus are $(q_{1,3}, q_{1,1})$ and $(q_{2,3}, q_{3,3})$. Whereas $\Pi_1 + \Pi_3$ is visualized in the left panel of Figure 4, the sum of the two 2-variety profits is visualized in the right panel of Figure 4. It is now easy to see that $\Pi_1 + \Pi_3 < 2\Pi_2$ because the two panels of Figure 4 are the same except that a “1” in the left panel is replaced by a “2” in the right panel.

The above logic is valid in general, and it is the intuition behind the proof of Theorem 4.

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Figure 3: Visualization of $\Pi_1 + \Pi_3$

Figure 4: Comparing $\Pi_1 + \Pi_3$ and $2\Pi_2$
Figure 5: The profit loss of deleting one variety

4.2 Rate of convergence results

This subsection provides our main rate of convergence result, which says that the marginal constrained profit $\Pi_n - \Pi_{n-1}$ is of order $n^{-3}$.

**Theorem 5 (Cubic rate result)** There exists a constant $C_0$ such that $n^3(\Pi_n - \Pi_{n-1}) \leq C_0$ for every $n = 1, 2, \ldots$.

Theorem 5 can be informally understood by looking at Figure 5, which demonstrates an example with $n = 4$. Starting from an optimal $n$-variety menu $(q_1, \cdots, q_n)$, if the monopoly deletes a variety with quality $q_i$, keeping the qualities of all other varieties unchanged, then the profit loss is the black area in Figure 5. Roughly speaking, this black area is proportional to $1/n^3$. It is because, as hinted by Figure 5, its horizontal length, vertical length, and the angles of its two ends are all proportional to $1/n$. We provide the complete proof in the following.

**Proof of Theorem 5.** Pick an arbitrary integer $n \geq 1$ and the associated $n$-variety constrained problem. Let $q_1, \cdots, q_n, t_1, \cdots, t_n$ be the optimal choices of this problem.

Now construct a suboptimal offering for a constrained problem with $n - 1$ varieties such that only a variety with quality $q_i$ is cancelled from the $n$-variety optimal offering, and the market originally served by this variety $q_i$ is served by the variety with quality $q_{i-1}$ (here $q_0$ is defined as 0, and $t_{n+1}$ as $\bar{t}$). In other words, $(q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_n; t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$
is offered. Denote the associated suboptimal profit as $\hat{\Pi}_{n-1}$. Then
\[
\Pi_n - \hat{\Pi}_{n-1} = \int_{t_i}^{t_{i+1}} [H(q_i, t) - H(q_{i-1}, t)] dF(t).
\]

Using the fact that $H(q_{i-1}, t_i) = H(q_i, t_i)$, the integrand can be written as follows:
\[
H(q_i, t) - H(q_{i-1}, t) = [H(q_i, t) - H(q_i, t_i)] - [H(q_{i-1}, t) - H(q_{i-1}, t_i)]
\]
\[
= \int_{t_i}^{q_i} \int_{t_i}^{t} H_{12}(q, x) dx dq.
\]

Note that $H_{12}(q, t) = J'(t)$ and $J'(\cdot)$ is continuous, so that there is a constant $\bar{H}_{12} \equiv \sup_{t \in [\tilde{t}, \bar{t}]} J'(t)$ such that $H_{12}(q, t) \leq \bar{H}_{12}$ for all $(q, t) \in [0, q(\tilde{t})] \times [t, \bar{t}]$. Thus we have
\[
\Pi_n - \hat{\Pi}_{n-1} \leq \int_{t_i}^{t_{i+1}} \int_{q_i}^{\bar{q}} \int_{t_i}^{t} \bar{H}_{12} dx dq dF(t)
\]
\[
\leq \bar{H}_{12} (t_{i+1} - t_i) (q_i - q_{i-1}) [F(t_{i+1}) - F(t_i)]
\]
\[
= \bar{H}_{12} (\bar{t} - t) q(\bar{t}) a_i b_i c_i
\]

where $a_i \equiv (t_{i+1} - t_i) / (\bar{t} - t)$, $b_i \equiv (q_i - q_{i-1}) / q(\bar{t})$ and $c_i \equiv F(t_{i+1}) - F(t_i)$.

Lemma 9, which is proved in Appendix, asserts that: If $a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_n \geq 0$, and $\sum_{i=1}^{n} a_i \leq 1$, $\sum_{i=1}^{n} b_i \leq 1$, $\sum_{i=1}^{n} c_i \leq 1$, then there exists some $i \in \{1, 2, \cdots, n\}$ such that $a_i b_i c_i \leq \frac{1}{n^n}$. From this lemma we get
\[
\Pi_n - \Pi_{n-1} \leq \Pi_n - \hat{\Pi}_{n-1} \leq \frac{\bar{H}_{12} (\bar{t} - t) q(\bar{t})}{n^3}.
\]

It proves the theorem with $C_0 \equiv \bar{H}_{12} (\bar{t} - t) q(\bar{t})$. □

Theorem 4 and Theorem 5 together tell us that the marginal constrained profit $\Pi_n - \Pi_{n-1}$ monotonically converges to its limit zero at the cubic rate $O(n^{-3})$, as $n \to \infty$.

Theorem 5 has yet another two corollaries. The first one is that the uncaptured profit $\Pi_\infty - \Pi_n$ is of order $n^{-2}$.

**Corollary 6 (Quadratic rate result)** There exists a constant $C_1$ such that $n^2(\Pi_\infty - \Pi_n) \leq C_1$ for every $n = 1, 2, \cdots$.

\[\text{[Footnote]}\text{The converse is not true, i.e. the quadratic rate result (Corollary 6) does not imply the cubic rate result (Theorem 5).}\]
Proof. Let $x_n \equiv \Pi_\infty - \Pi_n$. From Theorem 5, we have $0 \leq x_n - x_{n+1} \leq \frac{C_0}{(n+1)^3}$ for every $n$. Then

$$x_n = (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots \leq \frac{C_0}{(n+1)^3} + \frac{C_0}{(n+2)^3} + \cdots \leq C_0 \int_n^\infty \frac{1}{y^3} dy = \frac{C_0}{2n^2}.$$

It proves the corollary with $C_1 \equiv C_0/2$. ■

Thus, the constrained profit $\Pi_n$ monotonically converges to its limit $\Pi_\infty$ at the quadratic rate $O(n^{-2})$, as $n \to \infty$. This quadratic rate result can be understood by an insight shed from Figure 2: the uncaptured profit yielded from discrete offering is of second or higher order, but not of first order.

In a technical amendment in his book, Wilson (1993) also presents an “approximate optimality result” in the context of nonlinear pricing, in which a social planner uses $n$-part tariffs in a Ramsey pricing problem. His result is analogous to our Corollary 6 (for uncaptured profit $\Pi_\infty - \Pi_n$). In fact, the technique developed in this paper can be easily adapted to prove Wilson’s result.

Wilson (1993) does not provide results on the marginal benefit $\Pi_n - \Pi_{n-1}$ of increasing the number of options $n$ in the menu. In this paper, we are not only concerned with the approximate optimality of a short menu, but also concerned with the optimal choice of the menu. Thus, the marginal benefit of adding varieties (or options) is important, whose properties are provided by Theorem 4 and Theorem 5.

The second corollary of Theorem 5 is for the optimal number of varieties $n^*$. Recall that $n^*$ is the maximizer for the profit net of the cost for developing varieties. If the (per person) cost of developing every variety is some constant $k$, then the net (per person) profit to be maximized is $\Pi_n - nk$. Write the optimal number of varieties as a function of $k$, i.e. $n^*(k)$, then we have the following result.

**Corollary 7** There exists a constant $C_2$ such that $k^{1/3} n^*(k) \leq C_2$ for all $k > 0$.

**Proof.** If $k$ is too large so that $n^*(k) = 0$, the statement is trivially true. Suppose $k > 0$ is not so large. Since $\Pi_{n+1} - \Pi_n$ is decreasing in $n$ due to Theorem 4, $n^*$ is the unique integer

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8 In fact, such a fast convergence rate of $\Pi_n$ can be attained by a simple offering rule, which only involves uniformly distributed set of (suboptimal) varieties. Details are available upon request.

9 See Wilson (1993), Section 8.3 for details.
such that $\Pi_{n^*} - \Pi_{n^*-1} \geq k$ and $\Pi_{n^*+1} - \Pi_{n^*} < k$. Then

$$k^{1/3}n^* \leq (\Pi_{n^*} - \Pi_{n^*-1})^{1/3} n^* \leq (C_0)^{1/3}$$

where the last inequality is due to Theorem 5. It proves the corollary with $C_2 \equiv (C_0)^{1/3}$.

As the per person cost $k$ of developing a variety goes to zero, the optimal number of varieties $n^*(k)$ certainly goes to infinity. But Corollary 7 tells us that $n^*(k)$ goes to infinity at an extremely slow rate, in particular, $n^*(k) = O(k^{-1/3})$. It implies that, if the total (as opposed to per person) cost of developing a variety is fixed, and the number of potential buyers $N$ goes to infinity, $n^*$ goes to infinity at a very slow rate, namely, $O(N^{1/3})$.

## 5 A Solvable Class of Examples

The section studies a solvable class of examples, with uniform distribution and quadratic cost function. Its usefulness is two-fold. First, it demonstrates that the marginal constrained profit $(\Pi_{n+1} - \Pi_n)$ is small even for very small $n$ (say, $n = 3$). Second, it shows that the convergence rates we provided in the previous section are tight: one cannot prove a convergence rate faster than the cubic rate for $(\Pi_{n+1} - \Pi_n)$.

Consider the following class of particular cases. The distribution of buyers’ types is uniform, i.e. $t \sim U[\underline{t}, \bar{t}]$, so that for all $t \in [\underline{t}, \bar{t}]$

$$F(t) = \frac{t - \underline{t}}{\bar{t} - \underline{t}}$$

And the unit cost function $c(\cdot)$ is quadratic with the properties stated in (1), so that

$$c(q) = A_1q^2 + A_2q$$

where $A_1$ and $A_2$ are positive constants with $A_2 < \bar{t}$. The assumption $c'(0) \geq J(\bar{t})$ here is equivalent to $A_2 \geq 2\bar{t} - \bar{t}$.

As a result, the first-order difference equations (9) and (10) become linear as follows.

$$2A_1q_i + A_2 = t_i + t_{i+1} - \bar{t}, \quad (17)$$

$$A_1(q_i + q_{i-1}) + A_2 = 2t_i - \bar{t}. \quad (18)$$

With the two boundary conditions (11) and (12), the solutions of the monopolist’s optimal
discrete offering in the $n$-variety program have the following closed forms.

\[ q_i = \frac{(\bar{t} - A_2)i}{A_1(2n + 1)}, \quad (19) \]

\[ t_i = A_2 + \frac{n + i}{2n + 1} (\bar{t} - A_2). \quad (20) \]

Substitute them back into the $\Pi_n$ expression (8), we can explicitly express $\Pi_n$ in terms of $n$ and the parameters, namely,

\[ \Pi_n = \frac{n(n + 1)(\bar{t} - A_2)^3}{6A_1(\bar{t} - \bar{t})^2(2n + 1)^2}. \quad (21) \]

Its limit, $\Pi_\infty$, is

\[ \Pi_\infty = \frac{(\bar{t} - A_2)^3}{24A_1(\bar{t} - \bar{t})}. \quad (22) \]

Hence we have

\[ \frac{\Pi_n}{\Pi_\infty} = \frac{4n(n + 1)}{(2n + 1)^2} = 1 - \frac{1}{(2n + 1)^2}, \]

and

\[ \Pi_\infty - \Pi_n = \frac{1}{(2n + 1)^2} \Pi_\infty. \]

We can see that the fraction of constrained profit out of the unconstrained profit $\Pi_n/\Pi_\infty$ is approximately 89%, 96% or 98% for $n = 1, 2$ or 3 respectively. On the other hand, the marginal constrained profit $\Pi_{n+1} - \Pi_n$ is

\[ \Pi_{n+1} - \Pi_n = \frac{8(n + 1)}{(2n + 1)^2(2n + 3)^2} \Pi_\infty, \]

and the associated percentage change ($\Pi_{n+1} - \Pi_n)/\Pi_n$ is

\[ \frac{\Pi_{n+1} - \Pi_n}{\Pi_n} = \frac{2}{n(2n + 3)^2}. \]

We see that the percentage increase in profit due to adding one more variety $(\Pi_{n+1} - \Pi_n)/\Pi_n$ is approximately 8%, 2% or 0.8% for $n = 1, 2$ or 3 respectively.

Therefore, for the class of cases analyzed in this subsection, the fraction $\Pi_n/\Pi_\infty$ is really large and the percentage increase $(\Pi_{n+1} - \Pi_n)/\Pi_n$ is really small for small $n$.$^{10}$

$^{10}$The 2009 version of this paper also contains numerical simulations for other examples. In each of these examples $\Pi_3/\Pi_\infty$ is more than 97% and $(\Pi_4 - \Pi_3)/\Pi_3$ is less than 1%. See Wong (2009).
6 Conclusion

This paper studies how well a simple mechanism can approximate the fully optimal mechanism, and how the "complexity level" of a mechanism can be traded off with the "optimality level" of it, in the framework of Mussa and Rosen (1978) monopolistic quality differentiation.

We show that the marginal benefit of adding one more variety monotonically converges to zero at an extremely fast rate, namely the cubic rate $O(n^{-3})$. It suggests that the marginal benefit is small when the number of varieties $n$ is moderate. Furthermore, the numerical examples provided in the text suggest that the marginal benefit is small even for very small $n$ (say, $n = 3$). Therefore, lengthening the menu is usually not worthwhile. More generally speaking, simple mechanisms are justified against the theoretical complicated mechanisms.

There are at least three possible extensions we can consider. Firstly, we can consider endogenous costs of developing varieties which interact with the constrained program. Then one can ask if the small number of varieties is still robustly justified. Secondly, we might want to look at the problem in other market structures: oligopoly and monopolistic competition. It is interesting because under those market structures, each firm has incentive to differentiate its varieties against those of other competing firms. And under monopolistic competition, potential entrants would enter until the profitable opportunity vanishes. For these reasons, one might expect that the results presented in this paper would be weakened in those market structures. Thirdly, we might want to consider a more general multi-dimensional type situation. The solution pattern of the monopoly problem in such a situation is quite different from its single-dimensional counterpart studied here. Although the general treatment of this extended problem is much more difficult than the single-dimensional problem, it could be still tractable in terms of our current concerns. It is because the optimal solution needs not be fully treated for our purpose. Instead, one could play with some constructed suboptimal scheme, as in the proofs of Theorem 4 and Theorem 5.

Appendix

In order to prove Theorem 4, we firstly introduce a simple lemma.

**Lemma 8** Let $S_1$ and $S_2$ be two vectors of real numbers. (Their dimensions could be different.) Then

$$\max_{(2)}(S_1, S_2) \geq \min\{\max(S_1), \max(S_2)\},$$

where $\max_{(2)}(S)$ denotes the second largest element in $S$.

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11See Rochet and Chone (1998).
Proof. If \( \max(S_1, S_2) \) is in \( S_1 \), then \( \max(2)(S_1, S_2) \geq \max(S_2) \). If \( \max(S_1, S_2) \) is in \( S_2 \), then \( \max(2)(S_1, S_2) \geq \max(S_1) \). In both cases, our claim is true. ■

Proof of Theorem 4. Notice that

\[
\Pi_{n-1} = \int_{t_{1,n-1}}^{t_{2,n-1}} H(q_{1,n-1}, t)dF(t) + \int_{t_{2,n-1}}^{t_{3,n-1}} H(q_{2,n-1}, t)dF(t) + \cdots \\
+ \int_{t_{n-1,n-1}}^{t} H(q_{n-1,n-1}, t)dF(t),
\]

and that (see Figure 2) for all \( i, m \in \{1, 2, \cdots, n\} \) and all \( t \in [t_i, t_{i+1}] \), we have \( H(q_i, t) \geq H(q_m, t) \). Therefore

\[
\Pi_{n-1} = \int_{t}^{t} \max(0, H(q_{1,n-1}, t), H(q_{2,n-1}, t), \cdots, H(q_{n-1,n-1}, t)) dF(t).
\]

Now, for every \( t \) and \( n \) define the vector \( S_n(t) \) as

\[
S_n(t) \equiv (0, H(q_{1,n}, t), H(q_{2,n}, t), \cdots, H(q_{n,n}, t)).
\]

Then

\[
\Pi_{n-1} = \int_{t}^{t} \max(S_{n-1}(t))dF(t).
\]

Similarly,

\[
\Pi_{n+1} = \int_{t}^{t} \max(S_{n+1}(t))dF(t).
\]

Now, let us construct two suboptimal \( n \)-variety menus in the following way. First order the \( 2n \) numbers \( q_{1,n-1}, q_{2,n-1}, \cdots, q_{n-1,n-1}, q_{n-1,n+1}, q_{2,n+1}, \cdots, q_{n+1,n+1} \) ascendingly and denote the ordered numbers as \( q(1), q(2), \cdots, q(2n) \), where the subscript \( (i) \) denotes the \( i \)-th smallest number. Let us also define \( q(0) \) as 0.

Construct the first menu to include \( q(1), q(3), q(5), \cdots, q(2n-1) \) and the second menu to include \( q(2), q(4), q(6), \cdots, q(2n) \). And the corresponding \( t_i \)'s are constructed optimally given the quality offers, i.e.

\[
\frac{c(q_{(3)}) - c(q_{(1)})}{q_{(3)} - q_{(1)}} = J(t_1) \quad \text{or} \quad H(q_{(1)}, t_1) = H(q_{(3)}, t_1),
\]

\[
\vdots
\]

\[
\frac{c(q_{(2n-1)}) - c(q_{(2n-3)})}{q_{(2n-1)} - q_{(2n-3)}} = J(t_n) \quad \text{or} \quad H(q_{(2n-3)}, t_n) = H(q_{(2n-1)}, t_n)
\]
for the first menu and
\[
\frac{c(q_{(4)}) - c(q_{(2)})}{q_{(4)} - q_{(2)}} = J(t_1) \text{ or } H(q_{(2)}, t_1) = H(q_{(4)}, t_1),
\]
\[
\vdots
\]
\[
\frac{c(q_{(2n)}) - c(q_{(2n-1)})}{q_{(2n)} - q_{(2n-2)}} = J(t_n) \text{ or } H(q_{(2n-2)}, t_n) = H(q_{(2n)}, t_n)
\]
for the second menu.

Let \( \hat{\Pi}_n^{odd} \) and \( \hat{\Pi}_n^{even} \) be the corresponding profits for these two menus. Then
\[
\hat{\Pi}_n^{odd} + \hat{\Pi}_n^{even} = \int_{t_*}^{\hat{t}} \left[ \max \left( 0, H(q_{(1)}, t), H(q_{(3)}, t), \ldots, H(q_{(2n-1)}, t) \right) \\
+ \max \left( 0, H(q_{(2)}, t), H(q_{(4)}, t), \ldots, H(q_{(2n)}, t) \right) \right] dF(t).
\]

Notice that for any \( t \), if
\[
\max \left( 0, H(q_{(1)}, t), H(q_{(2)}, t), H(q_{(3)}, t), \ldots, H(q_{(2n)}, t) \right)
\]
is \( H(q_{(i)}, t) \), then \( \max(2) \left( 0, H(q_{(1)}, t), H(q_{(2)}, t), H(q_{(3)}, t), \ldots, H(q_{(2n)}, t) \right) \) must be either \( H(q_{(i-1)}, t) \) or \( H(q_{(i+1)}, t) \) (see Figure 2). Therefore,
\[
\hat{\Pi}_n^{odd} + \hat{\Pi}_n^{even} = \int_{t_*}^{\hat{t}} \left[ \max \left( 0, H(q_{(1)}, t), H(q_{(2)}, t), H(q_{(3)}, t), \ldots, H(q_{(2n)}, t) \right) \\
+ \max(2) \left( 0, H(q_{(1)}, t), H(q_{(2)}, t), H(q_{(3)}, t), \ldots, H(q_{(2n)}, t) \right) \right] dF(t)
\]
\[
= \int_{t_*}^{\hat{t}} \max(S_{n-1}(t), S_{n+1}(t)) dF(t) + \int_{t_*}^{\hat{t}} \max(2)(S_{n-1}(t), S_{n+1}(t)) dF(t).
\]

On the other hand,
\[
\Pi_{n-1} + \Pi_{n+1} = \int_{t_*}^{\hat{t}} \left[ \max(S_{n-1}(t)) + \max(S_{n+1}(t)) \right] dF(t)
\]
\[
= \int_{t_*}^{\hat{t}} \left[ \max \left\{ \max(S_{n-1}(t)), \max(S_{n+1}(t)) \right\} \\
+ \min \left\{ \max(S_{n-1}(t)), \max(S_{n+1}(t)) \right\} \right] dF(t)
\]
\[
= \int_{t_*}^{\hat{t}} \left[ \max(S_{n-1}(t), S_{n+1}(t)) + \min \left\{ \max(S_{n-1}(t)), \max(S_{n+1}(t)) \right\} \right] dF(t).
\]
Therefore,

\[
\left( \hat{\Pi}_n^{\text{odd}} + \hat{\Pi}_n^{\text{even}} \right) - (\Pi_{n-1} + \Pi_{n+1}) = \int_{t^*}^{t} \left[ \max(S_{n-1}(t), S_{n+1}(t)) - \min(\max(S_{n-1}(t)), \max(S_{n+1}(t))) \right] dF(t)
\]

\[\geq 0.\]

The last inequality is from Lemma 8.

Now, we have

\[2\Pi_n - (\Pi_{n-1} + \Pi_{n+1}) \geq \left( \hat{\Pi}_n^{\text{odd}} + \hat{\Pi}_n^{\text{even}} \right) - (\Pi_{n-1} + \Pi_{n+1}) \geq 0, \quad (A.1)\]

which implies \(\Pi_n - \Pi_{n-1} \geq \Pi_{n+1} - \Pi_n.\)

It remains to show that the two inequalities in (A.1) cannot be both equality. The second inequality in (A.1) is strict unless

\[\max(S_{n-1}(t), S_{n+1}(t)) = \min(\max(S_{n-1}(t)), \max(S_{n+1}(t)))\]

for almost all \(t.\)

By the definitions of \(S_{n-1}(t)\) and \(S_{n+1}(t)\) and noticing that \(S_{n+1}(t)\) has two more elements than \(S_{n-1}(t)\), the above cannot hold unless

\[
\begin{align*}
q_{1,n-1} &= q_{2,n+1} \\
q_{2,n-1} &= q_{3,n+1} \\
& \quad \vdots \\
q_{n-2,n-1} &= q_{n-1,n+1} \\
q_{n-1,n-1} &= q_{n,n+1}.
\end{align*}
\]

But given the above, the constructed varieties for \(\hat{\Pi}_n^{\text{odd}}\) and \(\hat{\Pi}_n^{\text{even}}\) are

\[
(q_{1,n+1}, q_{2,n+1}, \ldots, q_{n,n+1})
\]

and

\[
(q_{2,n+1}, q_{3,n+1}, \ldots, q_{n+1,n+1}) .
\]

Therefore, the two inequalities in (A.1) cannot be both equality unless there exist some optimal offers \(q_{1,n+1}, \ldots, q_{n+1,n+1}\) for the \((n + 1)\)-variety problem such that

(i) \(q_{1,n+1}, q_{2,n+1}, \ldots, q_{n,n+1}\) are some optimal offers for the \(n\)-variety problem;
(ii) \( q_{2,n+1}, q_{3,n+1}, \cdots, q_{n+1,n+1} \) are some optimal offers for the \( n \)-variety problem; and

(iii) \( q_{2,n+1}, q_{3,n+1}, \cdots, q_{n-1,n+1} \) are some optimal offers for the \((n-1)\)-variety problem.

However, checking our first-order conditions/difference equations (9) and (10), these are impossible. ■

The following lemma is used in the proof of Theorem 5.

**Lemma 9** If \( a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_n \geq 0 \), and \( \sum_{i=1}^{n} a_i \leq 1, \sum_{i=1}^{n} b_i \leq 1, \sum_{i=1}^{n} c_i \leq 1 \), then there exists some \( i \in \{1, 2, \cdots, n\} \) such that \( a_ib_ic_i \leq \frac{1}{n^3} \).

**Proof.** Consider the following maximization problem:

\[
\max_{a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_n} a_nb_n c_n \tag{A.2}
\]

subject to

\[
\sum_{i=1}^{n} a_i \leq 1, \sum_{i=1}^{n} b_i \leq 1, \sum_{i=1}^{n} c_i \leq 1,
\]

\[
a_i b_i c_i \geq \frac{1}{n^3} \quad \forall i = 1, 2, \cdots, n - 1.
\]

By Weierstrass Theorem, there is a global maximum. It suffices to prove that this global maximum is \( \frac{1}{n^3} \).

One can easily see that except the non-negativity constraints, all the other constraints must be binding. We can rewrite problem (A.2) as

\[
\max_{b_1, \cdots, b_{n-1}, c_1, \cdots, c_{n-1}} \left( 1 - \frac{1}{n^3 b_1 c_1} - \frac{1}{n^3 b_2 c_2} - \cdots - \frac{1}{n^3 b_{n-1} c_{n-1}} \right) (1-b_1-\cdots-b_{n-1})(1-c_1-\cdots-c_{n-1}).
\]

The first-order conditions are

\[
\left[ \frac{1}{n^3} \sum_{j=1}^{n-1} \frac{1}{b_j c_j} - 1 + \frac{1}{n^3 b_i^2 c_i} b_n \right] c_n = 0 \quad \forall i = 1, 2, \cdots, n - 1;
\]

\[
\left[ \frac{1}{n^3} \sum_{j=1}^{n-1} \frac{1}{b_j c_j} - 1 + \frac{1}{n^3 b_i^2 c_i} c_n \right] b_n = 0 \quad \forall i = 1, 2, \cdots, n - 1.
\]
At the optimum, neither \( b_n \) nor \( c_n \) can be zero, otherwise the objective function becomes zero as well. Therefore the first-order conditions imply

\[
\begin{align*}
  c_1 b_1^2 &= c_2 b_2^2 = \cdots = c_{n-1} b_{n-1}^2, \\
  b_1 c_1^2 &= b_2 c_2^2 = \cdots = b_{n-1} c_{n-1}^2.
\end{align*}
\]

Hence,

\[
\begin{align*}
  b_1 &= b_2 = \cdots = b_{n-1} \equiv \bar{b}, \\
  c_1 &= c_2 = \cdots = c_{n-1} \equiv \bar{c}.
\end{align*}
\]

Substituting them back into the first-order conditions, we get

\[
\begin{align*}
  b_1 &= b_2 = \cdots = b_n = c_1 = c_2 = \cdots = c_n = \frac{1}{n}.
\end{align*}
\]

Thus, the maximized value is \( \frac{1}{n^2} \), as desired. ■

References


Theory, 18(2), 301–317.

Behavior, 43(2), 214–238.

Screening,” Econometrica, 66(4), 783–826.

ROGERSON, W. P. (2003): “Simple menus of contracts in cost-based procurement and

