On Iterated Nash Bargaining Solutions*

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Abstract

This paper introduces a family of domains of bargaining problems allowing for non-convexity. For each domain in this family, single-valued bargaining solutions satisfying the Nash axioms are explicitly characterized as solutions of the iterated maximization of Nash products weighted by the row vectors of the associated bargaining weight matrices. This paper also introduces a simple procedure to standardize bargaining weight matrices for each solution into an equivalent triangular bargaining weight matrix, which is simplified and easy to use for applications. Furthermore, the standardized bargaining weight matrix can be recovered from bargaining solutions of simple problems. This recovering result provides an empirical framework for determining the bargaining weights.

Keywords: Bargaining problem, Non-convexity, Nash product, Iterated solution, Weight matrix

JEL Classification C71, C78

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1 Introduction

Bargaining theory in the spirit of the seminal work of Nash (1950, 1953) postulates that a group of players faces a set of feasible outcomes in the payoff space, and the implementation of an outcome requires unanimous agreement among the players. In cases of disagreement, the players end up getting some pre-specified outcome known as the disagreement point or the threat point. A bargaining problem in the sense of Nash is represented by a choice set along with a disagreement point in the payoff space.

In applications, the choice set may not be directly given but derived from more primitive data, and for this reason, it is often non-convex. Indeed, the non-convexity arises in bargaining problems including duopoly, employer-employee, and principal-agent bargaining problems.[1] The prevalence of the non-convexity raises serious theoretical and applied issues about the applicability of the Nash bargaining theory. For instance, there can be multiple Nash product maximizers for a non-convex problem. If single-valued solutions are selections of Nash product maximizers, which maximizers can be solutions satisfying the Nash axioms?

This paper concerns single-valued bargaining solutions allowing for non-convexity that satisfy the Nash axioms of Invariance to Affine Transformations (INV), Independence of Irrelevant Alternatives (IIA), and Strict Individual Rationality (SIR).[2] To increase the applicability to the extent possible, we consider various domains of

[1]The non-convexity of bargaining problems resulting from static duopolies with asymmetric constant marginal costs and concave monopoly profit functions has been well recognized in the industrial organization literature due to the work of Bishop (1960, p. 948), Schmalensee (1987, p. 354-356), Tirole (1988, p. 242, 271), and Qin et al. (2015), among others. In his analysis of sustainable collusive outcomes for a repeated duopoly with a fixed discount factor, Harrington (1991) found that as the choice set of the problem of bargaining over equilibrium selections, the set of payoff allocations that correspond to the set of subgame-perfect equilibrium outcomes, which are supportable by trigger strategies with Nash reversion, may be non-convex under the same primitives as those in the above papers (see Harrington (1991, p. 777)). As discovered in Aoki (1980), McDonald and Solow (1981), and Miyazaki (1984), non-convexity also arises in employer-employee bargaining problems. It is pointed out in Hougaard and Tvede (2003) (see also Ross (1973)) that bargaining problems arising from principal-agent moral hazard problems need not be convex unless randomized contracts are allowed.

[2]By a lemma in Roth (1977, p. 65), Pareto optimality is implied by INV, IIA, and SIR, and can therefore be replaced by SIR when INV and IIA are imposed. Since our domains of problems satisfy all the assumptions in Roth (1977) except for the convexity, we will consider INV, IIA, and SIR as Nash axioms.

bargaining problems that are characterized by certain conditions. Our derivation of bargaining solutions under the Nash axioms involves a sequence of n linearly independent vectors of weights (powers), where n is the number of players. Using the n vectors as the rows, a matrix forms which we refer to as an admissible weight matrix.

The first main result (Theorem 1) of this paper shows that for each of the domains, a single-valued solution satisfies INV, IIA, and SIR if and only if for some admissible weight matrix, it is found by an n-round iterative maximization of Nash products weighted by the row vectors of the admissible weight matrix. When the choice set of a bargaining problem is convex, the iteration terminates after the first round, in that the Nash product maximizer in the first round is unique which leads to a degenerate Nash product maximization for each of the remaining $n-1$ rounds. Hence, our result extends the Nash bargaining solution to allow for non-convexity.

Without further restrictions, two different admissible weight matrices may yield the same solution. Indeed, an equivalence relation between admissible weight matrices can be constructed using simple matrix transformation, such that equivalent matrices result in the same solution. We introduce an explicit procedure for standardizing admissible weight matrices into triangular ones (up to column permutation) with non-negative rows that sum to one, which we refer to as standardized bargaining weight matrix. We then show that equivalent matrices correspond (through standardization) to the same standardized bargaining weight matrix. Our standardization of admissible weight matrices with the non-convexity corresponds to the normalization of bargaining powers with the convexity. The standardization simplifies both the choice of an admissible weight matrix and computation of a solution.

Related literature. There have been both multi- and single-valued extensions of the Nash bargaining solution to allow for non-convexity. Since this paper concerns single-valued solutions, we refer the reader to Kaneko (1980), Herrero (1989), Peters and Vermeulen (2012), among others for details of multi-valued extensions. We focus on single-valued extensions in this selective literature review.

Conley and Wilkie (1996) established a single-valued extension which can be found via a two-step procedure. First, given a non-convex problem, convexify the choice set using randomized choices and consider the Nash solution for the convexified problem. Second, the intersection point of the original Pareto frontier with the segment between the disagreement point and the Nash solution for the convexified problem is the solution for the problem. For this extension, Conley and Wilkie showed that the solution can be characterized by replacing the IIA axiom in the collection of the Nash axioms by two other axioms, one being a Continuity axiom and another named as an Ethical Monotonicity axiom. Mariotti (1998) provided an alternative characterization of Conley and Wilkie's extension using a suitably restricted version of the IIA axiom, while maintaining the other axioms of Nash. A novelty of this alternative characterization is that it facilitates comparison with the standard axiom system and sheds light on the role played by the IIA axiom for this extension. The modifications of the Nash axioms required under these approaches make the extension fundamentally different from ours.

Qin et al. (2015) identified a domain of bargaining problems characterized by log-convexity and a regularity condition, to be referred to as the log-convex class. A problem is log-convex if the log transformation of the allocations of payoff gains relative to the disagreement point is strictly convex. Familiar examples of log-convex but not convex problems include duopoly bargaining problems with twice continuously differentiable demand functions, constant asymmetric marginal costs, and concave profit function for the more efficient firm. A result in Qin et al. (2015) states that the solution satisfying the Nash axioms of INV, IIA, and SIR on the log-convex class of problems is unique up to choices of bargaining powers or, equivalently, up to solutions for the Divide-the-Dollar problem. The log-convex class is tight in the sense that the result is no longer valid for any domain of problems containing but not equal to the log-convex class.

Most related to our paper are Zhou (1997) and Peters and Vermeulen (2012). Zhou (1997) considered single-valued solutions on the domain of comprehensive (relative to the disagreement point) problems. This domain is included in the family of admissible domains we consider in this paper. Zhou imposed the INV axiom on the subdomain of convex problems only, while we impose it on the entire domain of bargaining problems. With the INV axiom imposed on the subdomain of convex problems, Nash product maximizers can all be realizable as Nash solutions for the problem. In this sense, Zhou's solutions are loose, which stands in sharp contrast to the much more tightly structured solutions in our paper.

As an auxiliary result for our practitioner friendly representation of single-valued solutions satisfying the Nash axioms in Theorem 1, we first show that single-valued bargaining solutions satisfying the Nash axioms can be found by the iterated maximization of Nash products weighted by the row vectors of suitable bargaining weight matrices. Peters and Vermeulen (2012) studied multi-valued solutions. It turns out that their analysis can be adapted to establish our auxiliary result. To avoid excessively sophisticated methods of proofs in an attempt to help the reader to gain better understanding of the result, we offer a simpler and more elementary method to establish the iterative derivation of single-valued solutions. Our elementary method also helps to uncover more structural details of the solutions. For example, in the two-person case, there are exactly two single-valued solutions, each given by the (dictatorial) selection of the first-round Nash product maximizers in accordance with the maximization of one player's payoff. See Qin et al. (2019) and Remark 2 in the current paper for details.

One particular type of applications of axiomatic bargaining theory is to fit empirical data with bargaining solutions (see, e.g., Shimer (2005)). Unless the symmetry axiom is imposed, the bargaining weights of a solution are not explicitly specified under the Nash axioms.[3] Thus, it is desirable to recover the bargaining weights associated with a solution using simple problems, such as the Divide-the-Dollar problem, for which suitable solutions for the corresponding applications can be easily and meaningfully specified. It is well known that for convex problems, such recoverability can be achieved through the solution of the Divide-the-Dollar problem alone. The second main result (Theorem 2) of this paper shows that the admissible weight matrices of solutions of non-convex problems can also be recovered through an iterative procedure involving solutions of the Divide-the-Dollar problem and its subproblems.

The rest of the paper is organized as follows. Section 2 presents the preliminaries we work with in this paper. Section 3 and Section 4 present results related to the derivation, standardization, and recoverability of admissible weight matrices of single-valued solutions satisfying Nash axioms. Section 5 concludes. Appendices A

[3]Notice that the symmetry axiom is no longer as natural for a problem when solutions for the problem are not unique. When the symmetry axiom is removed, Kalai (1977) showed that an asymmetric Nash solution, unique up to the specification of a solution for *Divide-the-Dollar problem*, is characterized by the remaining axioms on the class of compact convex problems. A payoff allocation for Divide-the-Dollar problem has been customarily interpreted as representing players' bargaining powers (this interpretation can be traced to Shubik (1959, p. 50)). Given players' bargaining powers, the asymmetric Nash solution assigns to each problem the unique maximizer of the generalized Nash product—the Nash product weighted by the bargaining powers.

and B cover proofs of results not proven in the main text.

2 Preliminaries

Let $n \geq 2$ denote the number of players. A (n-player) bargaining problem is composed of a compact set $S \subseteq \mathbb{R}^n$ of feasible payoff allocations the players can jointly achieve with agreement, and a disagreement point (or threat point) $d \in S$ the players end up getting in case of disagreement, such that d is strictly Pareto dominated in S (i.e., $u \gg d$ for some $u \in S$). A bargaining problem (S, d) is *comprehensive* (relative to the disagreement point) if $u \in S$ and $d \leq v \leq u$ imply $v \in S$. A *positive affine transformation* for player i is a mapping $\tau_i : \mathbb{R} \to \mathbb{R}$ such that for some real numbers $a_i > 0$ and b_i, $\tau_i(u_i) = a_i u_i + b_i$ for all $u_i \in \mathbb{R}$. Given $\tau_1, \ldots, \tau_n$, set $\tau(u) = (\tau_1(u_1), \ldots, \tau_n(u_n))$ for all $u \in \mathbb{R}^n$.

A *(single-valued) bargaining solution* on a domain $\mathcal{B}$ of bargaining problems is a mapping $f : \mathcal{B} \to \mathbb{R}^n$ such that $f(S, d) \in S$ for all $(S, d) \in \mathcal{B}$. We allow significant flexibility of the domains. Specifically, a domain $\mathcal{B}$ is regarded as admissible if it satisfies the following properties.

B1: $\mathcal{B}$ is closed under positive affine transformation: for any $(S, d) \in \mathcal{B}$ and positive affine transformations $\tau : \mathbb{R}^n \to \mathbb{R}^n$, $(\tau(S), \tau(d)) \in \mathcal{B}$.

B2: $\mathcal{B}$ is closed under individual rationality restriction: $(S, d) \in \mathcal{B}$ and $T = \{u \in S : u \geq d\}$ imply $(T, d) \in \mathcal{B}$.

B3: $\mathcal{B}$ includes all comprehensive problems consisting of only individually rational choices: $(S, d) \in \mathcal{B}$ whenever (S, d) is comprehensive (relative to the disagreement point) and $S - d \subseteq \mathbb{R}^n_+$.[4]

Property B1 is one which would be satisfied if for each player, the set of equivalent utility transformations includes positive affine transformations. This is a very mild condition. Property B2 is natural because individually irrational choices are usually regarded as irrelevant. Property B3 imposes a large size restriction on the domain of bargaining problems. This restriction ensures that the domain is rich enough for

[4]We use $S - d$ to denote the set $\{u - d : u \in S\}$.

the solutions under the Nash axioms to be found through an iterative maximization of Nash products.

We now state the Nash axioms for a bargaining solution f on a given domain $\mathcal{B}$ to satisfy.

Invariance to Positive Affine Transformations (INV): For any $(S, d) \in \mathcal{B}$ and positive affine transformations $\tau : \mathbb{R}^n \to \mathbb{R}^n$, $f(\tau(S), \tau(d)) = \tau(f(S, d))$.

Independence of Irrelevant Alternatives (IIA): For any $(S, d), (T, d) \in \mathcal{B}$ with $S \subseteq T$, $f(T, d) \in S$ implies $f(S, d) = f(T, d)$.

Strict Individual Rationality (SIR): For any $(S, d) \in \mathcal{B}$, $f(S, d) \gg d$.

These axioms are now standard.[5] Roth (1977) showed that Pareto optimality is implied by INV, IIA, and SIR for the domain of convex problems. As shown in Lemma A.1 in Appendix A of the present paper, this result also holds for domains characterized by B1–B3.

3 Bargaining solutions satisfying Nash axioms

3.1 Main result

Let $\Delta^n \equiv \{\alpha \in \mathbb{R}^n : \sum_{i=1}^n |\alpha_i| = 1\}$, $\Delta^n_+ \equiv \Delta^n \cap \mathbb{R}^n_+$, and $\Delta^n_{++} \equiv \Delta^n \cap \mathbb{R}^n_{++}$. A square matrix is *upper triangular* if all its entries below the main diagonal are zero. Our characterization of bargaining solutions satisfying the aforementioned Nash axioms hinges on "weight matrices" with specific properties.

Definition 1. A $n \times n$ real matrix is said to be an *admissible weight matrix* if it has full rank and its first row is in $\mathbb{R}^n_{++}$. An admissible weight matrix is said to be a *triangularly standard weight matrix* if its row vectors belong to Δ^n_+ and it is upper triangular up to permutation of columns.

We are interested in solutions satisfying the Nash axioms on the domains satisfying B1–B3. As will be shown shortly, the solutions are exactly those that can be computed via an n-round iterative maximization of generalized Nash products defined below.

[5] We refer to Roth (1979) and Moulin (1988) among others for discussions.

Definition 2. Given any $n \times n$ matrix $\mathbf{W}$, with the j-th row denoted by α^j, a bargaining solution f on a domain $\mathcal{B}$ is said to be a $\mathbf{W}$*-bargaining solution* if

$$f(S,d) \in \Sigma^n(S,d) \text{ for any } (S,d) \in \mathcal{B}, \tag{1}$$

where $\Sigma^n(S,d)$ is the set of maximizers of the last round of iterative maximization:

$$\Sigma^j(S,d) \equiv \underset{u \in \Sigma^{j-1}(S,d)}{\operatorname{argmax}} \prod_{i=1}^{n} (u_i - d_i)^{\alpha_i^j}, \;\; j = 1, \ldots, n, \tag{2}$$

$$\Sigma^0(S,d) \equiv S \cap \{u \in \mathbb{R}^n : u \geq d\}.$$

Due to the non-convexity, bargaining solutions satisfying the Nash axioms are not unique. Nonetheless, they can be represented as $\mathbf{W}$-bargaining solutions. This representation is established in our main result given below.

Theorem 1. *Let $\mathcal{B}$ be any domain of n-player bargaining problems satisfying B1–B3.*

(a) For any triangularly standard weight matrix $\mathbf{W}$, there is a unique $\mathbf{W}$-bargaining solution on domain $\mathcal{B}$; moreover, this solution satisfies INV, IIA, and SIR.

(b) Conversely, for any bargaining solution f on $\mathcal{B}$ satisfying INV, IIA, and SIR, there is a unique triangularly standard weight matrix $\mathbf{W}$ such that f is the $\mathbf{W}$-bargaining solution.

Before proving Theorem 1, two remarks on some particular implications of the result are in order.

Remark 1. The two parts of Theorem 1 together establish a one-to-one correspondence between the set of bargaining solutions satisfying the Nash axioms and the set of triangularly standard weight matrices.

Remark 2. When $n = 2$, Theorem 1 says that bargaining solutions satisfying the Nash axioms are dictatorial selections of generalized Nash product maximizers (since the second row of a 2×2 triangularly standard weight matrix is either $(1,0)$ or $(0,1)$).[6]

By Remark 2, any "non-polar" Nash product maximizer for 2-player problems can never be selected by solutions satisfying the Nash axioms. The example below provides an illustration.

[6] Qin et al. (2019) show that these solutions are implementable as unique subgame perfect equilibrium payoff allocations of a sequential game.

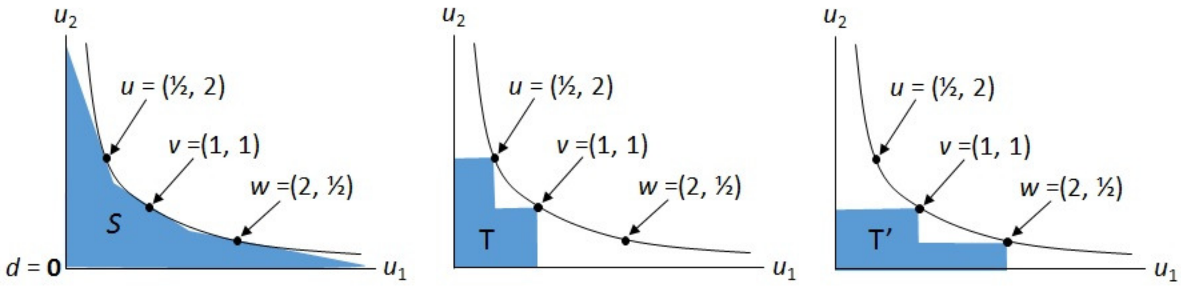

Figure 1: An illustration of Example 1

Example 1. Suppose $n = 2$ and $f(S^\circ, \mathbf{0}) = (\frac{1}{2}, \frac{1}{2})$, where $S^\circ \equiv \{u \in \mathbb{R}^2_+ : u_1 + u_2 \leq 1\}$, so that the Nash product is symmetric between the two players. Let $S = \{u \in \mathbb{R}^2_+ : \min\{2u_1 + \frac{1}{2}u_2, u_1 + u_2, \frac{1}{2}u_1 + 2u_2\} \leq 2\}$. Then, the bargaining problem $(S, \mathbf{0})$ has three Nash product maximizers, which are $u = (\frac{1}{2}, 2)$, $v = (1, 1)$, and $w = (2, \frac{1}{2})$. However, $f(S, \mathbf{0}) = v$ is impossible if f satisfies the Nash axioms. To see this, suppose $f(S, \mathbf{0}) = v$. Let T be the comprehensive hull of $\{u, v\}$ and T' the comprehensive hull of $\{v, w\}$, as indicated in Figure 1. Then IIA implies $f(T, \mathbf{0}) = f(T', \mathbf{0}) = v$. However, by INV, $f(T, \mathbf{0}) = v$ implies $f(T', \mathbf{0}) = w$, which contradicts $f(T', \mathbf{0}) = v$.

3.2 Proof of Theorem 1(a)

Part (a) of Theorem 1 is relatively straightforward. In fact, this part relies on only a small part of the assumptions. Hence, in this subsection we state and prove a stronger version of Theorem 1(a).

Proposition 1. *Let $\mathcal{B}$ be any domain of n-player bargaining problems (not necessarily satisfying B1–B3). For any admissible weight matrix $\mathbf{W}$, there is a unique $\mathbf{W}$-bargaining solution on domain $\mathcal{B}$; moreover, this solution satisfies INV, IIA, and SIR.*

Proof. Consider any bargaining problem (S, d) and any admissible weight matrix $\mathbf{W}$ with the j-th row denoted by α^j. Let $\Sigma^j(S, d)$ $(j = 0, \ldots, n)$ be as defined in Definition 2. Since S is compact and d is strictly Pareto dominated in S, $\Sigma^0(S, d)$ is nonempty and compact. Since $\alpha^1 \gg \mathbf{0}$, we have $\Sigma^1(S, d) - d \subseteq \mathbb{R}^n_{++}$. Since

$\Sigma^n(S,d) \subseteq \cdots \subseteq \Sigma^1(S,d)$, we also have $\Sigma^j(S,d) - d \subseteq \mathbb{R}^n_{++}$ for $j = 1, \ldots, n$. Let

$$g^j(u,d) \equiv \prod_{i=1}^{n} (u_i - d_i)^{\alpha_i^j}, \quad j = 1, \ldots, n.$$

Since $g^j(\cdot, d)$ for each j is continuous, it follows from the Maximum Theorem that $\Sigma^j(S,d)$ for each j is nonempty and compact. We next claim that $\Sigma^n(S,d)$ is a singleton. Take $u, v \in \Sigma^n(S,d)$. Then, by construction, $g^j(u,d) = g^j(v,d)$ for each $j = 1, \ldots, n$. Taking log, it follows that

$$\mathbf{W} \cdot [\log(u-d) - \log(v-d)] = \mathbf{0},$$

where $\log(u-d)$ denotes the column vector $[\log(u_i - d_i)]_{i=1,\ldots,n}$ and similarly for $\log(v-d)$. Since $\mathbf{W}$ has full rank, we must have $u = v$. Therefore, $\Sigma^n(S,d)$ must be a singleton. The first claim in the proposition follows.

To see the second claim, observe that, for any positive affine transformations τ, $\Sigma^j(\tau(S), \tau(d)) = \tau(\Sigma^j(S,d))$; thus INV holds. Also, since $\alpha^1 \gg \mathbf{0}$, we have $\Sigma^n(S,d) \gg d$; thus SIR holds. Finally, since f is given by a process of iterated maximization, IIA holds. ■

3.3 Proof of Theorem 1(b)

To prove Theorem 1(b), we first prove that any bargaining solution satisfying the Nash axioms is the $\mathbf{W}$-bargaining solution for some admissible weight matrix $\mathbf{W}$. Then we will standardize the weight matrix into a triangularly standard one to establish uniqueness.

Proposition 2. *Let $\mathcal{B}$ be any domain of n-player bargaining problems satisfying B1–B3. A bargaining solution f on $\mathcal{B}$ satisfies INV, IIA, and SIR only if it is the $\mathbf{W}$-bargaining solution for some admissible weight matrix $\mathbf{W}$.*[7] *Moreover, the row vectors of $\mathbf{W}$ can be uniquely chosen to be pairwise orthogonal from Δ^n.*

Proposition 2 can be regarded as a variant of Theorem 5.4 in Peters and Vermeulen (2012).[8] As mentioned in the Introduction, we provide an elementary proof

[7] The converse ("if" part) is in Proposition 1.
[8] See also Naumova and Yanovskaya (2001).

in Appendix A.

Different admissible weight matrices may result in identical bargaining solutions through the iterative procedure described in (1) and (2). We now introduce an equivalence relation between admissible weight matrices and then show that all weight matrices in the same equivalence class yield the same bargaining solution.

Definition 3. Two admissible matrices are said to be *equivalent* if one can be transformed into the other through a finite sequence of following operations:[9]

(i) multiplying a row by a positive scalar;

(ii) replacing a row by the sum of itself and a scalar multiple of a preceding row.

It is worth noticing that the set of row operations in Definition 3 is a proper subset of those involved in Gaussian elimination in linear algebra.[10]

Lemma 1. *Let* $\mathbf{W}, \tilde{\mathbf{W}}$ *be admissible weight matrices. Then, the* $\mathbf{W}$*- and* $\tilde{\mathbf{W}}$*-bargaining solutions are identical if and only if* $\mathbf{W}$ *and* $\tilde{\mathbf{W}}$ *are equivalent.*

By Proposition 2 and Lemma 1, each bargaining solution satisfying the Nash axioms can be identified with an equivalence class of admissible weight matrices as specified in Definition 3. Proposition 2 also shows that orthogonalization and row rescaling provide one way to uniquely standardize the weight matrices for each solution. However, orthogonalization necessarily generates negative bargaining weights in non-first rounds, which is unappealing in respect of economic interpretations. To establish the alternative (and more appealing) standardization given in Theorem 1 (i.e., transforming to be triangularly standard), we first establish the following result, which is proved in Appendix B.

Proposition 3. *For any admissible weight matrix* $\mathbf{W}$*, there is a unique triangularly standard weight matrix that is equivalent to* $\mathbf{W}$*.*

We are now ready to prove Theorem 1(b).

[9] It is clear that this definition does define an equivalence relation (i.e., reflexive, symmetric, and transitive binary relation) over the space of admissible weight matrices.

[10] The row operations involved in Gaussian elimination are: swapping two rows, multiplying a row by a non-zero scalar, and replacing a row by the sum of itself and a scalar multiple of another row.

Proof of Theorem 1(b). Suppose that f on $\mathcal{B}$ satisfies INV, IIA, and SIR. From Proposition 2, there is a unique admissible weight matrix $\mathbf{W}$ with pairwise orthogonal row vectors in Δ^n such that f is the $\mathbf{W}$-bargaining solution. By Proposition 3, there is a unique triangularly standard weight matrix $\mathbf{W}^*$ that is equivalent to $\mathbf{W}$. Thus, it follows from Lemma 1 that f is the $\mathbf{W}^*$-bargaining solution. ■

Remark 3. As by-products, the proofs of Lemma 1 and Proposition 3 also establish constructive procedures to standardize weight matrices. Specifically, to standardize weight matrices so as to have pairwise orthogonal row vectors in Δ^n, we can apply the Gram-Schmidt orthogonalization process and row rescaling as we do in the proof of the "only if" part of Lemma 1. To standardize the weight matrices into a triangularly standard one as in Theorem 1, we can apply the procedure in the proof of the existence part of Proposition 3. The following examples illustrate this procedure, which exhibits some similarities with Gaussian elimination.[11]

Example 2. Let $n = 3$. The first matrix in (3) below is an admissible weight matrix and we want to transform it into an equivalent triangularly standard weight matrix through the row operations in Definition 3.

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & 5 & 1 \\ 6 & -3 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 0 \\ 6 & -3 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 0 \\ 3 & -6 & 0 \end{bmatrix}$$
$$\Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 0 \\ 21/2 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 5/9 & 4/9 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{3}$$

The major part of the transformation is to obtain an upper triangular (up to permutations of columns) matrix with non-negative entries. Once this part is done, one can easily rescale each row using operation (i) in Definition 3 to obtain a triangularly standard weight matrix. To obtain the upper triangularity structure, we first need to eliminate one entry (i.e., create zero entry) of the second row. Since all entries in the first row are positive as guaranteed by the admissibility, operation (ii) in Definition

[11] Although the purpose of Gaussian elimination is, like ours, to obtain triangular structure, it does not guarantee non-negativity. Also recall that not all the row operations needed in Gaussian elimination are allowed in Definition 3.

3 allows us to eliminate any single entry of the second row. But if we choose to eliminate the first or the second entry, some other entry of the row would become negative. So we can only eliminate the third entry, by adding -1 times the first row into the second row. Then we obtain the second matrix in (3). Triangularity then requires the third entry of the third row must also be eliminated, so we need to add -3 times the first row into the third row, resulting in the third matrix in (3). To eliminate one more entry of the third row, as required by the triangularity, we need to use the second row to eliminate either the first or the second entry of the third row. But eliminating the first entry would result in a negative entry of the row. So we can only eliminate the second entry, by adding $3/2$ times the second row into the third row, resulting in the fourth matrix in (3). Finally, we multiply each row by a positive scalar to make its entries sum to one. Then the result, the last matrix in (3), is guaranteed to be triangularly standard.

Example 3. Let $n = 3$. The first matrix in (4) is an admissible weight matrix and we want to transform it into an equivalent triangularly standard weight matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & -2 \\ 1 & -3 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 5 & 0 & 0 \\ 1 & -3 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 5 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$
$$\Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 5 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4)$$

Like in the previous example, we first use the first row to eliminate one entry of the second row, at the same time avoiding negative entries. So we add 2 times the first row into the second row, resulting in the second matrix in (4). Unlike in the previous example, now there are two zero entries in the second row. So in the next stage should we use the first row to eliminate the second or the third entry of the third row? If we choose to eliminate the third entry (i.e., add 2 times the first row into the third row), the third row would become $(3, -1, 0)$; but then the best we can further do is to use the second row to eliminate the first entry of the third row, leaving the second entry negative; the non-negativity could not be fulfilled. Therefore, starting from the second matrix in (4), we have to use the first row to eliminate the second entry of the third row, by adding 3 times the first row into the third row, resulting in the third matrix in (4). Then, after using the second row to eliminate the first entry of the third row, we obtain the fourth matrix in (4), which has all non-negative entries. Finally, we multiply each row by a positive scalar to make its entries sum to one. The result, the last matrix in (4), is triangularly standard.

4 Recoverability of weight matrices

If f is a bargaining solution satisfying Nash axioms on a domain of *convex* bargaining problems, all one needs to characterize f is an n-dimensional weight vector (the first row of an admissible weight matrix that characterizes f). It is well known that such a weight vector can be recovered from $f(S^\circ, \mathbf{0})$, where $S^\circ \equiv \{u \in \mathbb{R}^n_+ : \sum_{i=1}^n u_i \leq 1\}$ and $(S^\circ, \mathbf{0})$ is known as the Divide-the-Dollar problem.

It is worth noting that the Nash axioms without that of symmetry cannot pin down which weight vector to use for computing the solution. Nonetheless, due to its simplicity with the convexity, suitable solutions for the Divide-the-Dollar problem are relatively easier to specify for given applications. Hence, the recoverability of the weight vector from solution of $(S^\circ, \mathbf{0})$ is desirable and important.

For each of the domains of bargaining problems satisfying B1–B3, it is also the case that the Nash axioms of INV, IIA, and SIR are not decisive as to which admissible weight matrix should be adopted to (iteratively) compute the solution. Thus, as in the case with convexity, it would be desirable to be able to recover the weight matrix using simple problems, such as the Divide-the-Dollar problem. Our result below shows that there is a sequential procedure involving the Divide-the-Dollar problem and its subproblems through which an admissible weight matrix of a given solution can be recovered. Once such a matrix is recovered, one may standardize it into a triangularly standard one following the procedure described in Remark 3 and illustrated in Examples 2 and 3.

Theorem 2. *Let $\mathcal{B}$ be a domain of n-player bargaining problems satisfying B1–B3, and f be a bargaining solution on $\mathcal{B}$ satisfying INV, IIA, and SIR. For $\phi \in (0, 1/n)$, recursively define*

$$S^1 \equiv S^\circ,$$

$$\beta^1 \equiv f(S^\circ, \mathbf{0}),$$
$$S^j \equiv \left\{ u \in S^{j-1} : \prod_{i=1}^n u_i^{\beta_i^{j-1}} = \phi \right\}, \quad j = 2, \ldots, n,$$
$$\beta^j \equiv f(\mathrm{com}(S^j), \mathbf{0}), \quad j = 2, \ldots, n,$$

where $\mathrm{com}(S^j) \equiv (S^j - \mathbb{R}^n_+) \cap \mathbb{R}^n_+$ *denotes the comprehensive hull of* S^j*. Also define* $\hat{\mathbf{W}}$ *as the* $n \times n$ *matrix with* j*-th row being* β^j*. Then* $\hat{\mathbf{W}}$ *is an admissible weight matrix and* f *is the* $\hat{\mathbf{W}}$*-bargaining solution.*

An important issue in applications of bargaining theory is how to fit empirical data with axiomatic bargaining solutions. Hypothesizing that the data corresponds to a single-valued Nash bargaining solution, one would need to identify a corresponding weight matrix. Theorem 2 specifies a method for systematically recovering the weight matrix from a given solution.

5 Conclusion

The Nash bargaining theory has become one of the most fruitful paradigms in game theory and has inspired a large literature on axiomatic solutions and applications. Our paper contributes to this vast literature in three ways. First, to enhance applicability, we introduce a family of domains of bargaining problems allowing for non-convexity. Second, using elementary methods, we completely characterized the explicit structures of the solutions and associated bargaining powers under the Nash axioms for each of the domains. The approach with elementary methods simplifies applications of the results. Third, due to the structural complexity, the recoverability of the bargaining powers for non-convex problems under the Nash axioms is not as transparent as for convex problems. We introduced a method to systematically recover bargaining powers from given solutions of simpler problems.

Appendix

A Proof of Proposition 2

To prove Proposition 2, in the following we fix a domain $\mathcal{B}$ satisfying B1–B3 and a bargaining solution f on $\mathcal{B}$.

For $S \subseteq \mathbb{R}^n$, we let $\text{Par}(S) \equiv \{u \in S : \nexists v \in S \text{ s.t. } v \geq u \text{ and } v \neq u\}$ denote the strict Pareto frontier of S and $\text{com}(S) \equiv (S - \mathbb{R}^n_+) \cap \mathbb{R}^n_+$ the comprehensive hull of S. When S is finite such as $S = \{u, v, w\}$, we may write $\text{Par}(\{u, v, w\})$ and $\text{com}(\{u, v, w\})$ as $\text{Par}(u, v, w)$ and $\text{com}(u, v, w)$. Note that, by B3, $(\text{com}(S), \mathbf{0}) \in \mathcal{B}$.

Lemma A.1. *If f satisfies INV, IIA, and SIR, then, for any $(S, d) \in \mathcal{B}$, $f(S, d) \in \text{Par}(S)$.*

Proof. By B1 and INV, we assume $d = \mathbf{0}$ without loss of generality. Set $S_+ \equiv S \cap \mathbb{R}^n_+$. Since $S_+ \subseteq S$, $\text{Par}(S_+) \subseteq \text{Par}(S)$. By B2, $(S_+, \mathbf{0}) \in \mathcal{B}$ and by SIR, $f(S, \mathbf{0}) \in S_+$. Hence, by IIA, $f(S, \mathbf{0}) = f(S_+, \mathbf{0})$.

Let $T \equiv \text{com}(S_+)$. Then $S_+ \subseteq T \subseteq \mathbb{R}^n_+$, $\text{Par}(T) = \text{Par}(S_+)$, and $(T, \mathbf{0}) \in \mathcal{B}$ by B3. Suppose $v \equiv f(T, \mathbf{0}) \notin \text{Par}(T)$. Then there exists $w \in T$ such that $w \geq v$ and $w \neq v$. Since $v \gg \mathbf{0}$ by SIR, we also have $w \gg \mathbf{0}$. Set $\beta_i \equiv v_i / w_i > 0$ for $i = 1, \ldots, n$. Then $\beta \equiv (\beta_1, \ldots, \beta_n) \leq (1, \ldots, 1)$ and $\beta \neq (1, \ldots, 1)$. Let $T' \equiv \{(\beta_1 u_1, \ldots, \beta_n u_n) : u \in T\}$. By B1, $(T', \mathbf{0}) \in \mathcal{B}$. By construction $T' \subseteq T$ and $f(T, \mathbf{0}) = v \in T'$. Consequently, by IIA, $f(T', \mathbf{0}) = v$. However, by INV, $f(T', \mathbf{0}) = (\beta_1 v_1, \ldots, \beta_n v_n) \neq v$, a contradiction. Therefore, $f(T, \mathbf{0}) \in \text{Par}(T)$.

Finally, since $S_+ \subseteq T$ and $f(T, \mathbf{0}) \in \text{Par}(T) = \text{Par}(S_+) \subseteq S_+$, IIA implies $f(S_+, \mathbf{0}) = f(T, \mathbf{0})$. In summary, $f(S, \mathbf{0}) = f(S_+, \mathbf{0}) = f(T, \mathbf{0}) \in \text{Par}(T) = \text{Par}(S_+) \subseteq \text{Par}(S)$. ■

Define a binary relation $\succ_f$ on $\mathbb{R}^n_{++}$ based on f as follows:

$$u \succ_f v \text{ iff } u \neq v \text{ and } f(\text{com}(u, v), \mathbf{0}) = u.$$

Lemma A.2. *Suppose f satisfies INV, IIA, and SIR.*

(a) $\succ_f$ is a strongly monotone strict linear order; that is, for any $u, v, w \in \mathbb{R}^n_{++}$, (i) exactly one of $u \succ_f v$, $v \succ_f u$, and $u = v$ holds, (ii) $u \succ_f v$ and $v \succ_f w$ imply

$u \succ_f w$, and (iii) $u \geq v$ and $u \neq v$ imply $u \succ_f v$.

(b) For any $(S, \mathbf{0}) \in \mathcal{B}$, $f(S, \mathbf{0})$ is the unique maximizer of $\succ_f$ on $S \cap \mathbb{R}^n_{++}$; that is, $f(S, \mathbf{0}) \in S \cap \mathbb{R}^n_{++}$ and

$$f(S, \mathbf{0}) \succ_f u \text{ for any } u \in (S \cap \mathbb{R}^n_{++}) \backslash \{f(S, \mathbf{0})\}.$$

Proof. Throughout this proof f is fixed and we drop the subscript "f" in $\succ_f$.

We first establish part (a). Property (i) is obvious and property (iii) follows from Lemma A.1. To show property (ii), suppose $u \succ v$ and $v \succ w$. Then property (i) implies $u \neq w$. Again by property (i), it suffices to show that $w \succ u$ is false. Suppose on the contrary $w \succ u$. By Lemma A.1, $f(\mathrm{com}(u, v, w), \mathbf{0}) \in \{u, v, w\}$. If $f(\mathrm{com}(u, v, w), \mathbf{0}) = u$, then IIA implies $u \succ w$, contradicting $w \succ u$. Similarly, $f(\mathrm{com}(u, v, w), \mathbf{0}) = v$ contradicts $u \succ v$ and $f(\mathrm{com}(u, v, w), \mathbf{0}) = w$ contradicts $v \succ w$. Therefore, we must have $u \succ w$ and hence property (ii).

We next establish part (b). Set $S_{++} \equiv S \cap \mathbb{R}^n_{++}$ and $S_+ \equiv S \cap \mathbb{R}^n_+$. That $f(S, \mathbf{0}) \in S_{++}$ directly follows from SIR. Set $u^* \equiv f(S, \mathbf{0})$. We need to show $u^* \succ u$, or equivalently, $u^* = f(\mathrm{com}(u, u^*), \mathbf{0})$, for any $u \in S_{++} \backslash \{u^*\}$. By B2, $(S_+, \mathbf{0}) \in \mathcal{B}$. By IIA, $u^* = f(S_+, \mathbf{0})$. By B3, $(\mathrm{com}(S_+), \mathbf{0}) \in \mathcal{B}$ and $(\mathrm{com}(u, u^*), \mathbf{0}) \in \mathcal{B}$. By Lemma A.1, we have $f(\mathrm{com}(S_+), \mathbf{0}) \in S_+$. Thus, by IIA, we have $f(\mathrm{com}(S_+), \mathbf{0}) = f(S_+, \mathbf{0}) = u^* \in \mathrm{com}(u, u^*)$. By IIA again, $f(\mathrm{com}(S_+), \mathbf{0}) = f(\mathrm{com}(u, u^*), \mathbf{0})$. This shows $u^* = f(\mathrm{com}(u, u^*), \mathbf{0})$. ■

Let $\log u \equiv (\log u_1, \ldots, \log u_n)$ for $u \in \mathbb{R}^n_{++}$; let $\log S \equiv \{\log u : u \in S\}$ for $S \subseteq \mathbb{R}^n_{++}$.

Lemma A.3. *Suppose f satisfies INV, IIA, and SIR. For any $u \in \mathbb{R}^n_{++}$ define*

$$H_f(u) \equiv \left\{v \in \mathbb{R}^n_{++} : v \succ_f u\right\},$$

$$L_f(u) \equiv \left\{v \in \mathbb{R}^n_{++} : u \succ_f v\right\}.$$

(a) $\log H_f(u) = \log H_f(\mathbf{1}) + \log u$*;*
(b) $\log L_f(u) = \log L_f(\mathbf{1}) + \log u$*;*
(c) $\log H_f(\mathbf{1}) = -\log L_f(\mathbf{1})$*;*
(d) $\log H_f(\mathbf{1})$ *and* $\log L_f(\mathbf{1})$ *are convex.*

Proof. Throughout this proof f is fixed and we drop the subscript "f" in $\succ_f$, $H_f(\cdot)$, and $L_f(\cdot)$.

Step 1. We claim that, for any $\lambda \in \mathbb{R}$ and $u^0, u^1, v^0, v^1 \in \mathbb{R}^n_{++}$ with $u^1 \succ u^0$ and $\log v^1 - \log v^0 = \lambda(\log u^1 - \log u^0)$, we have $v^1 \succ v^0$ if $\lambda > 0$ and $v^0 \succ v^1$ if $\lambda < 0$.

To see this, first define u^t for $t \in \mathbb{R}$ by

$$\log u^t \equiv \log u^0 + t\left(\log u^1 - \log u^0\right).$$

Substep 1.1. We show that the claim of this step holds for $\lambda = 1$ and $\lambda = -1$. When $\lambda = 1$, $\log v^1 - \log v^0 = \log u^1 - \log u^0$, or equivalently $v_i^1/u_i^1 = v_i^0/u_i^0$, for each $i \in \{1, \dots, n\}$. Thus, by INV and $u^1 \succ u^0$,

$$v^1 = \left(\frac{v_i^1}{u_i^1}u_i^1\right)_{i=1,\dots,n} = \left(\frac{v_i^0}{u_i^0}u_i^1\right)_{i=1,\dots,n} \succ \left(\frac{v_i^0}{u_i^0}u_i^0\right)_{i=1,\dots,n} = v^0.$$

For the case of $\lambda = -1$, the claim can be similarly proved with the roles of v^0 and v^1 interchanged.

Substep 1.2. We show that the claim of this step holds when $\lambda = 1/m$ for any positive integer m. By the assumptions on u^0, u^1, v^0, v^1 and Substep 1.1, it suffices to show $u^{1/m} \succ u^0$. Suppose on the contrary $u^0 \succ u^{1/m}$. Since $\log u^{(i-1)/m} - \log u^{i/m}$ for $i = 1, 2, \dots, m$ are all equal to $\frac{1}{m}(\log u^0 - \log u^1)$, it follows from Substep 1.1 that $u^0 \succ u^{1/m} \succ \cdots \succ u^{(m-1)/m} \succ u^1$. The transitivity of $\succ$ established in Lemma A.2(a) then implies that $u^0 \succ u^1$, which contradicts $u^1 \succ u^0$.

Substep 1.3. We show that the claim of this step holds for all $\lambda > 0$. Given that $\lambda > 0$, it follows from the assumptions on u^0, u^1, v^0, v^1 and Substep 1.1 that it suffices to show $u^\lambda \succ u^0$. Consider the "log-convex hull" of $\{u^0, u^\lambda\}$ defined by $T = \{u^t : 0 \le t \le \lambda\}$. From B3, $(\mathrm{com}(T), \mathbf{0}) \in \mathcal{B}$. Suppose on the contrary $f(\mathrm{com}(T), \mathbf{0}) = u^t$ for some $t \in [0, \lambda)$. Then we can pick a large enough positive integer m such that $t + \frac{1}{m} < \lambda$. By construction, $u^{t+1/m} \in T$ and by IIA, $f(\mathrm{com}(u^t, u^{t+1/m}), \mathbf{0}) = u^t$. It follows that $u^t \succ u^{t+1/m}$. Hence, from Substep 1.1 it follows that $u^0 \succ u^{1/m}$. This contradicts Substep 1.2. Therefore, $f(\mathrm{com}(T), \mathbf{0}) \neq u^t$ for any $t \in [0, \lambda)$. But $f(\mathrm{com}(T), \mathbf{0}) \in T$ from Lemma A.1. Consequently, $f(\mathrm{com}(T), \mathbf{0}) = u^\lambda$. Thus, by IIA, $f(\mathrm{com}(u^0, u^\lambda), \mathbf{0}) = u^\lambda$ or equivalently, $u^\lambda \succ u^0$.

Substep 1.4. Substep 1.1 with $\lambda = -1$ and Substep 1.3 together imply that the claim of this step also holds for all $\lambda < 0$.

Step 2. Establish parts (a), (b), and (c).

The result in Step 1 with $\lambda = 1$ can be written as: if $\log v^1 - \log v^0 = \log u^1 - \log u^0$, then $v^1 \succ v^0$ and $u^1 \succ u^0$ are equivalent; $v^0 \succ v^1$ and $u^0 \succ u^1$ are equivalent. Since $\log \mathbf{1} = \mathbf{0}$, parts (a) and (b) follow. The result in Step 1 with $\lambda = -1$ can be written as: if $\log v^1 - \log v^0 = -(\log u^1 - \log u^0)$, then $v^1 \succ v^0$ and $u^0 \succ u^1$ are equivalent; $v^0 \succ v^1$ and $u^1 \succ u^0$ are equivalent. Part (c) follows.

Step 3. Establish part (d).

We only prove $\log L(u)$ is convex; the proof for $\log H(u)$ is analogous. Suppose that $\log v^0$ and $\log v^1$ are in $\log L(u)$ or equivalently, $u \succ v^0$ and $u \succ v^1$. Consider $\log v^\lambda = \lambda \log v^1 + (1-\lambda) \log v^0$ for $\lambda \in (0,1)$. We need to show that $u \succ v^\lambda$. If $v^0 = v^1$, then $u \succ v^0 = v^\lambda$. If $v^0 \succ v^1$, then Step 1 implies $u \succ v^0 \succ v^\lambda$. If $v^1 \succ v^0$, then Step 1 implies $u \succ v^1 \succ v^\lambda$. This shows $u \succ v^\lambda$. ■

Lemma A.4. *Suppose f satisfies INV, IIA, and SIR. Define $H_f(\cdot)$ and $L_f(\cdot)$ as in Lemma A.3. There exist pairwise orthogonal non-zero vectors $\alpha^1, \ldots, \alpha^n \in \mathbb{R}^n$ with $\alpha^1 \geq \mathbf{0}$ such that $x \in \log H_f(\mathbf{1})$ (resp. $x \in \log L_f(\mathbf{1})$) if and only if there is some $j \in \{1, \ldots, n\}$ such that $\alpha^j \cdot x > 0$ (resp. $\alpha^j \cdot x < 0$) and $\alpha^k \cdot x = 0$ for $k = 1, \ldots, j-1$. Moreover, every such α^i $(i = 1, \ldots, n)$ is unique up to positive scalar multiplication.*

Proof. For notational convenience, let $\hat{H} \equiv \log H_f(\mathbf{1})$ and $\hat{L} \equiv \log L_f(\mathbf{1})$. By Lemma A.2(a), $\hat{H} \cap \hat{L} = \varnothing$, $\hat{H} \cup \hat{L} = \mathbb{R}^n \backslash \{\mathbf{0}\}$, and $\mathbb{R}^n_{++} \subseteq \hat{H}$. By Lemma A.3(c), $\hat{H} = -\hat{L}$. By Lemma A.3(d), $\hat{H}, \hat{L}$ are convex.

Note that $\hat{H}$ and $\hat{L}$ are nonempty. Denote their common boundary by E^1. Since $\hat{H}$ and $\hat{L}$ are convex, E^1 is a hyperplane with some non-zero normal vector $\alpha^1 \in \mathbb{R}^n$. Since $\hat{H} = -\hat{L}$, we have $\mathbf{0} \in E^1$. This concludes that E^1 is an $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$. Choose the direction of the normal vector α^1 so that $\{x : \alpha^1 \cdot x > 0\} \subseteq \hat{H}$ and $\{x : \alpha^1 \cdot x < 0\} \subseteq \hat{L}$. Since $\mathbb{R}^n_{++} \subseteq \hat{H}$, we have $\alpha^1 \geq 0$.

Since E^1 is a nontrivial vector space, $\hat{H} \cup \hat{L} = \mathbb{R}^n \backslash \{\mathbf{0}\}$, $\hat{H} = -\hat{L}$, and $\hat{H}, \hat{L}, E^1$ are all convex, it follows that $\hat{H} \cap E^1$ and $\hat{L} \cap E^1$ are nonempty, convex, and they share the common boundary $E^2 \subseteq E^1$ with some non-zero normal vector $\alpha^2 \in E^1$. Applying the previous reasoning, E^2 is an $(n-2)$-dimensional vector subspace of $\mathbb{R}^n$. Choose the direction of the normal vector α^2 so that $\{x : \alpha^2 \cdot x > 0\} \cap E^1 \subseteq \hat{H}$ and $\{x : \alpha^2 \cdot x < 0\} \cap E^1 \subseteq \hat{L}$. Since $\alpha^2 \in E^1$ and $\alpha^1 \perp E^1$, we have $\alpha^2 \perp \alpha^1$.

Repeating the preceding process and reasoning, a collection of subspaces $\{E^1, \ldots, E^n\}$ and a collection $\{\alpha^1, \ldots, \alpha^n\}$ of corresponding normal vectors can be generated, such that $E^1 \supset E^2 \supset \cdots \supset E^n$, each E^j is an $(n-j)$-dimensional vector subspace of $\mathbb{R}^n$ so that $E^n = \{\mathbf{0}\}$ and each normal vector α^j of E^j satisfies $\{x : \alpha^j \cdot x > 0\} \cap E^{j-1} \subseteq \hat{H}$ and $\{x : \alpha^j \cdot x < 0\} \cap E^{j-1} \subseteq \hat{L}$ (we let $E^0 \equiv \mathbb{R}^n$). The process stops at $E^n = \{\mathbf{0}\}$ because $\hat{H} \cap E^n$ and $\hat{L} \cap E^n$ are empty. Hence, the separation between $\hat{H}$ and $\hat{L}$ is complete, in that given any point $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, we can determine whether $x \in \hat{H}$ or $x \in \hat{L}$ using $\{\alpha^1, \ldots, \alpha^n\}$.

It is easy to verify that $\hat{H}$ and $\hat{L}$ are exactly described as in the lemma. Furthermore, the direction of each α^j is uniquely determined. It remains to show that $\alpha^1, \ldots, \alpha^n$ are pairwise orthogonal. To this end, pick any $j, k \in \{1, \ldots, n\}$ with $j < k$. By our construction, $\alpha^k \in E^{k-1} \subseteq E^j$ and $\alpha^j \perp E^j$. It follows that $\alpha^j \perp \alpha^k$. ■

Proof of Proposition 2. Suppose f satisfies INV, IIA, and SIR. Let $\alpha^1, \ldots, \alpha^n \in \mathbb{R}^n$ be vectors with the properties claimed in Lemma A.4. Let $\mathbf{W}$ be the $n \times n$ matrix with i-th row being α^i. Since $\alpha^1, \ldots, \alpha^n$ are pairwise orthogonal and non-zero, they are linearly independent.

By Lemma A.3(b), $\log v \in \log L_f(u)$ if and only if for some $j \in \{1, \ldots, n\}$, $\alpha^j \cdot (\log v - \log u) < 0$ and $\alpha^k \cdot (\log v - \log u) = 0$ for $k < j$. Equivalently, $u \succ_f v$ if and only if there is some $j \in \{1, \ldots, n\}$ such that $g^j(v, \mathbf{0}) < g^j(u, \mathbf{0})$ and $g^k(v, \mathbf{0}) = g^k(u, \mathbf{0})$ for $k < j$, where $g^j(u, d) \equiv \prod_{i=1}^n (u_i - d_i)^{\alpha_i^j}$ for $u \geq d$ and $j = 1, \ldots, n$. Now, Lemma A.2(b) implies that f is the $\mathbf{W}$-bargaining solution.

Notice that α^1 is the unique maximizer of $g^1(\cdot, \mathbf{0})$ on S°. Hence, $\alpha^1 = f(S^\circ, \mathbf{0})$. By SIR, $\alpha^1 \gg \mathbf{0}$. Therefore, $\mathbf{W}$ is an admissible weight matrix. From Lemma A.4 we know the $\alpha^1, \ldots, \alpha^n$ above can be uniquely normalized to be in Δ^n. ■

B Other proofs not in the text

Proof of Lemma 1. "If" part. The fact that any row operation of the first kind does not change the represented solution is rather obvious. Now let us consider a row operation of the second kind. Consider replacing the j-th row α^j by $\alpha^j + \theta\alpha^k$, where θ is a scalar (not necessarily positive) and α^k is a preceding row (i.e., $1 \leq k < j$). From (2), all elements in $\Sigma^{j-1}(S, d)$ must have a common value of $\prod_{i=1}^n (u_i - d_i)^{\alpha_i^k}$. This common value, denoted as π, must be positive. Therefore, the new maximization problem in the j-th round can be written as

$$\max_{u \in \Sigma^{j-1}(S,d)} \prod_{i=1}^{n} (u_i - d_i)^{\alpha_i^j + \theta \alpha_i^k} = \max_{u \in \Sigma^{j-1}(S,d)} \pi^{\theta} \prod_{i=1}^{n} (u_i - d_i)^{\alpha_i^j},$$

which is clearly equivalent to the original maximization problem in the j-th round.

"Only if" part. First, clearly the row operations in this lemma are reversible; that is, if $\tilde{\mathbf{W}}$ can be obtained from $\mathbf{W}$ through an application of those row operations, then $\mathbf{W}$ can also be obtained from $\tilde{\mathbf{W}}$ through an application of those row operations. Also note that our row operations of the second kind are the operations needed in Gram-Schmidt orthogonalization process. Therefore, a finite sequence of our row operations allows us to transform $\mathbf{W}$ through Gram-Schmidt orthogonalization process into some $n \times n$ matrix whose rows are pairwise orthogonal. This new matrix has full rank because $\mathbf{W}$ has. Therefore, each row of the new matrix must be non-zero and hence we can apply our row operations of the first kind to normalize each row to be in Δ^n. Thus, we know that a finite sequence of our row operations can transform $\mathbf{W}$ into some $n \times n$ matrix $\mathbf{A}$ whose rows are pairwise orthogonal and in Δ^n. Similarly, $\tilde{\mathbf{W}}$ can also be transformed into some $n \times n$ matrix $\tilde{\mathbf{A}}$ with the same properties through a finite sequence of our row operations. From the "if" part we proved above, we know that f is the $\mathbf{A}$-bargaining solution and $\tilde{f}$ is the $\tilde{\mathbf{A}}$-bargaining solution. Now, if it is the case that $f = \tilde{f}$, then from the uniqueness part of Proposition 2(b) we know that $\mathbf{A} = \tilde{\mathbf{A}}$. Therefore, we can from $\mathbf{W}$ obtain $\mathbf{A}$ and then from $\mathbf{A}$ obtain $\tilde{\mathbf{W}}$ through a finite sequence of our row operations. ■

Proof of Proposition 3. Existence. Let $\mathbf{W}$ be an admissible weight matrix (i.e., is $n \times n$, of full rank, and with first row in $\mathbb{R}^n_{++}$). Let $\mathcal{M}^k$ denote the set of $k \times k$ full rank matrices with its first row belonging to Δ^k_+ and the first non-zero entry in every column being positive. First, $\mathbf{W}$ is equivalent to some matrix in $\mathcal{M}^n$ because the first row of $\mathbf{W}$ is in $\mathbb{R}^n_{++}$ and hence can be multiplied by a positive scalar to become in Δ^n_{++}. In the following, we shall show that, for any positive integer k, if $\mathbf{B} \in \mathcal{M}^{k+1}$, then $\mathbf{B}$ is equivalent to some matrix $\mathbf{D} \in \mathcal{M}^{k+1}$ such that, for some $i^* \in \{1, \ldots, k+1\}$, (i) the i^*-th column of $\mathbf{D}$ has all entries being zero except the first entry (which must be positive), and (ii) the submatrix of $\mathbf{D}$ after deleting the first row and i^*-th column of $\mathbf{D}$ belongs to $\mathcal{M}^k$. ($\mathbf{B}$ and $\mathbf{D}$ must also have the same first row since they are equivalent and both are in $\mathcal{M}^{k+1}$.) Once the last claim is proved, it is easy to see that we can iteratively transform $\mathbf{W}$ into equivalent matrices such that an equivalent triangularly standard weight matrix is ultimately obtained.

Now, let k be a positive integer and $\mathbf{B} \in \mathcal{M}^{k+1}$ with its (j,i)-th entry denoted as β_i^j. Recursively define

$$K^1 \equiv \left\{ i \in \{1, \ldots, k+1\} : \beta_i^1 > 0 \right\},$$

$$K^j \equiv \operatorname*{argmin}_{i \in K^{j-1}} \frac{\beta_i^j}{\beta_i^1} \text{ for } j = 2, \ldots, k+1.$$

Note that the above sets are nonempty because K^1 is nonempty (since $\beta^1 \in \Delta_+^{k+1}$). Note also that K^{k+1} must be a singleton (otherwise $\mathbf{B}$ must not have full rank). Now, let i^* be the unique element of K^{k+1}. Then we use the first row of $\mathbf{B}$ to eliminate the i^*-th entry in all other rows. That is, we replace each j-th row β^j $(j = 2, \ldots, k+1)$ by $\beta^j - (\beta_{i^*}^j / \beta_{i^*}^1)\beta^1$. After the above row operations, the second row must be non-negative and non-zero. Then we multiply the second row by a positive constant to make it belonging to Δ_+^k. Now, the resulting matrix is our desired matrix $\mathbf{D}$.

Uniqueness. It suffices to show that, if $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ full rank matrices which have all rows in Δ_+^n and are upper triangular up to permutations of columns, then $\mathbf{A}$ and $\mathbf{B}$ are equivalent only when they are equal. In the rest of this proof, we show the last claim by a proof by contradiction.

Take any $\mathbf{A}, \mathbf{B}$ that have the above properties. Suppose on the contrary that $\mathbf{A}$ and $\mathbf{B}$ are equivalent but $\mathbf{A} \neq \mathbf{B}$. Let α^j denote the j-th row of $\mathbf{A}$, and β^j the j-th row of $\mathbf{B}$. Since $\mathbf{A} \neq \mathbf{B}$, there is a unique row index $k \in \{1, \ldots, n\}$ such that

$$\alpha^k \neq \beta^k, \tag{B.1}$$

$$\alpha^j = \beta^j \text{ whenever } j < k. \tag{B.2}$$

Since all rows in $\mathbf{A}$ and $\mathbf{B}$ are in Δ_+^n, we have

$$\alpha^j, \beta^j \in \Delta_+^n \text{ for } j = 1, \ldots, n. \tag{B.3}$$

Note that any upper triangular matrix has full rank if and only if all of its diagonal entries are non-zero. Therefore, any $n \times n$ full rank matrix that is upper triangular up to permutations of columns has only one permuting of columns that can make it upper triangular. To save one notation, we without loss of generality assume that this permuting for $\mathbf{A}$ is the identity permuting (i.e., $\mathbf{A}$ itself is upper triangular). Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ denote the permuting of columns that makes $\mathbf{B}$ upper triangular. Thus,

$$\alpha_i^j = \beta_{\sigma(i)}^j = 0 \text{ whenever } i < j. \tag{B.4}$$

Since $\mathbf{A}$ and $\mathbf{B}$ are of full rank, we have

$$\alpha_i^i, \beta_{\sigma(i)}^i > 0 \text{ for } i = 1, \ldots, n. \tag{B.5}$$

Since $\mathbf{A}$ and $\mathbf{B}$ are equivalent, we know from our definition of matrix equivalence that there exist (unique) constants $\theta^1, \ldots, \theta^k$ such that

$$\theta^1 \alpha^1 + \cdots + \theta^k \alpha^k = \beta^k \tag{B.6}$$

and

$$\theta^k > 0. \tag{B.7}$$

Now, if it is the case that $\theta^j = 0$ whenever $j < k$, then (B.6) becomes $\theta^k \alpha^k = \beta^k$, which together with (B.3) implies

$$\theta^k = \theta^k \sum_{i=1}^n \alpha_i^k = \sum_{i=1}^n \beta_i^k = 1,$$

so that (B.6) becomes $\alpha^k = \beta^k$, contradicting (B.1). Therefore, $k \geq 2$ and at least one of $\theta^1, \ldots, \theta^{k-1}$ is non-zero. Then there is a unique index $k' \in \{1, \ldots, k-1\}$ such that

$$\theta^{k'} \neq 0, \tag{B.8}$$

$$\theta^j = 0 \text{ whenever } j < k'. \tag{B.9}$$

Using (B.9) to simplify (B.6), we have

$$\theta^{k'}\alpha^{k'} + \cdots + \theta^k\alpha^k = \beta^k. \tag{B.10}$$

Considering the k'-th entry of (B.10) and using (B.4) to simplify, we have

$$\theta^{k'}\alpha^{k'}_{k'} = \beta^k_{k'}. \tag{B.11}$$

Since we have $\alpha^{k'}_{k'} > 0$ from (B.5) and $\beta^k_{k'} \geq 0$ from (B.3), it follows from (B.8) and (B.11) that

$$\theta^{k'} > 0. \tag{B.12}$$

We are now ready to derive a contradiction as follows:

$$\begin{aligned}
0 &= \beta^k_{\sigma(k')} \\
&= \theta^{k'}\alpha^{k'}_{\sigma(k')} + \cdots + \theta^k\alpha^k_{\sigma(k')} \\
&= \theta^{k'}\beta^{k'}_{\sigma(k')} + \cdots + \theta^{k-1}\beta^{k-1}_{\sigma(k')} + \theta^k\alpha^k_{\sigma(k')} \\
&= \theta^{k'}\beta^{k'}_{\sigma(k')} + \theta^k\alpha^k_{\sigma(k')} \\
&> \theta^k\alpha^k_{\sigma(k')} \\
&\geq 0.
\end{aligned}$$

In the above chain, the first and fourth lines are from (B.4); the second and third lines are from (B.10) and (B.2) respectively; the second last line is from (B.5) and (B.12); the last line is from (B.7) and (B.3). ■

Proof of Theorem 2. By Theorem 1(b) or Proposition 2, f is the $\mathbf{W}$-bargaining solution for some admissible weight matrix $\mathbf{W}$. Let α^i $(i = 1, \ldots, n)$ denote the i-th row of $\mathbf{W}$. Then $\alpha^1, \ldots, \alpha^n \in \mathbb{R}^n$ are linearly independent, $\alpha^1 \gg \mathbf{0}$, and (1) is satisfied. We shall show that for $j = 1, \ldots, n$,

(i) S^j is nonempty,

(ii) $\beta^j \in \Delta^n_{++}$,

(iii) there are constants $\theta^1, \ldots, \theta^j$ such that $\theta^1\alpha^1 + \cdots + \theta^j\alpha^j = \beta^j$ and $\theta^j > 0$,

(iv) $\beta^1, \dots, \beta^j$ are linearly independent.

Once (i)–(iv) are established, we would know that $\hat{\mathbf{W}}$ is an admissible weight matrix that is equivalent to $\mathbf{W}$; hence from Lemma 1 f is the $\hat{\mathbf{W}}$-bargaining solution and the proof would be completed.

We prove (i)–(iv) by induction. Observe that claims (i)–(iv) hold for $j = 1$. Take any $k \in \{2, \dots, n\}$ and suppose (i)–(iv) hold for $j = 1, \dots, k-1$. Then,

$$S^k = \left\{ u \in \mathbb{R}^n_{++} : \sum_{i=1}^n u_i \leq 1 \text{ and } \sum_{i=1}^n \beta_i^j \ln u_i = \ln \phi \text{ for } j = 1, \dots, k-1 \right\}.$$

The log-transformation of S^k can be written as $\ln S^k = V_1 \cap V_2$ where

$$V_1 \equiv \left\{ \ln u \in \mathbb{R}^n : \sum_{i=1}^n u_i \leq 1 \right\},$$

$$V_2 \equiv \left\{ \ln u \in \mathbb{R}^n : \sum_{i=1}^n \beta_i^j \ln u_i = \ln \phi \text{ for } j = 1, \dots, k-1 \right\}.$$

Since (iv) holds for $j = k-1$ (i.e., $\beta^1, \dots, \beta^{k-1}$ are linearly independent), V_2 is an $(n-k+1)$-dimensional flat surface in $\mathbb{R}^n$ and hence, it is nonempty. On the boundary of V_1, there are many points above V_2 and many below V_2. Indeed, for any $u \in \Delta^n_{++}$ close enough to $(1/n, \dots, 1/n)$, it follows from $\beta^1, \dots, \beta^{k-1} \in \Delta^n_{++}$ and $\phi \in (0, 1/n)$ that $\ln u$ lies above V_2; for any $u \in \Delta^n_{++}$ that has at least one coordinate close enough to 0, $\ln u$ lies below V_2. This shows that $\ln S^k$ is nonempty. Consequently, S^k is nonempty and thus (i) holds for $j = k$.

Since (iii) holds for $j = 1, \dots, k-1$, from the proof of the "if" part of Lemma 1 it follows that the sets of maximizers in the first $k-1$ rounds of the iterative maximization in Definition 2 do not change if we replace $\alpha^1, \dots, \alpha^{k-1}$ with $\beta^1, \dots, \beta^{k-1}$. Therefore, $\Sigma^{k-1}(\text{com}(S^k), \mathbf{0}) = S^k$. The k-th round of the iterative maximization can be written as

$$\max_{u \in \mathbb{R}^n_{++}} \sum_{i=1}^n \alpha_i^k \ln u_i \text{ s.t. } \sum_{i=1}^n u_i \leq 1 \text{ and } \sum_{i=1}^n \beta_i^j \ln u_i = \ln \phi, \; j = 1, \dots, k-1. \quad \text{(B.13)}$$

Considering $\hat{u} \equiv \ln u$, (B.13) is equivalent to

$$\max_{\hat{u}\in\mathbb{R}^n} \sum_{i=1}^{n} \alpha_i^k \hat{u}_i \text{ s.t. } \sum_{i=1}^{n} e^{\hat{u}_i} \leq 1 \text{ and } \sum_{i=1}^{n} \beta_i^j \hat{u}_i = \ln\phi,\ j = 1, \ldots, k-1. \tag{B.14}$$

Problem (B.14) is a convex optimization problem. Moreover, the constraints satisfy the Slater condition because $(\ln\phi, \ldots, \ln\phi)$ satisfies the affine equality constraints (since (ii) holds for $j = 1, \ldots, k-1$) and strictly satisfies the inequality constraint (since $\phi < 1/n$).

By definition, β^k must be a solution of problem (B.13) and hence, $\ln\beta^k$ must be a solution of problem (B.14). Thus, the following Kuhn-Tucker conditions for problem (B.14) hold:

$$\alpha_i^k - \lambda\beta_i^k - \sum_{j=1}^{k-1} \mu^j \beta_i^j = 0 \text{ for } i = 1, \ldots, n, \tag{B.15}$$

$$\lambda \geq 0,\quad \sum_{i=1}^{n} \beta_i^k \leq 1,\quad \lambda\left(1 - \sum_{i=1}^{n} \beta_i^k\right) = 0, \tag{B.16}$$

$$\sum_{i=1}^{n} \beta_i^j \ln\beta_i^k = \ln\phi \text{ for } j = 1, \ldots, k-1,$$

where λ and $\mu^1, \ldots, \mu^{k-1}$ are the Langrangian multipliers. If $\lambda = 0$, then (B.15) implies that α^k is a linear combination of $\beta^1, \ldots, \beta^{k-1}$ and hence, it is also a linear combination of $\alpha^1, \ldots, \alpha^{k-1}$ because (iii) holds for $j = 1, \ldots, k-1$. This contradicts the linear independence of $\alpha^1, \ldots, \alpha^n$. Therefore, it must be the case that $\lambda > 0$. From (B.16), $\sum_{i=1}^n \beta_i^k = 1$, which implies that (ii) holds for $j = k$. Moreover, from (B.15),

$$\beta_i^k = \frac{1}{\lambda}\left(\alpha_i^k - \sum_{j=1}^{k-1} \mu^j \beta_i^j\right).$$

This establishes (iii) for $j = k$.

Finally, the validity of claim (iii) for $j = 1, \ldots, k$ together with the linear independence of $\alpha^1, \ldots, \alpha^k$ implies that $\beta^1, \ldots, \beta^k$ are linearly independent. It follows that claim (iv) holds for $j = k$. ■

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